E209A LECTURE 3

GOALS OF THIS LECTURE:

- introduce some notation from Real Analysis which we'll use
- introduce the concept of Index Theory
- Poincaré Theorem: relationship between nodes, centers, foci, saddles

a closed orbit.

REFS:

SASTRY § 2.4
KHALIL § 2.6
Analysis of Planar Dynamical Systems (cont.)

Some definitions about sets in $\mathbb{R}^n$:

- set $E \subset \mathbb{R}^n$ (subset of)
- point $p \in \mathbb{R}^n$ (element of)

**Def** A **neighborhood** of $p$ is a set $N(p, S) \subset \mathbb{R}^n$ consisting of all points $q$ such that $||p-q|| < S$ for $S > 0$.

**Def** A point $p$ is a **limit point** of the set $E$ if every neighborhood of $p$ contains a point $q \neq p$ such that $q \in E$.

**Def** $E$ is **closed** if every limit point of $E$ is a point of $E$.

**Def** $E$ is **open** if every point $p \in E$ has a neighborhood $N(p, S) \subset E$.

**Def** $E$ is **bounded** if there is a real number $M \in \mathbb{R}$ and a point $q \in \mathbb{R}^n$ such that $||p-q|| < M$ for all $p \in E$. 

E.g. here $p$ is on the boundary of $E$, and is thus a limit pt. of $E$. 

$E$ is closed $\iff$ $E$ contains its boundary.

$E$ is open $\iff$ $E$ does not contain its boundary.
Index Theory:

Defn let $J$ be a closed, positively oriented contour in $\mathbb{R}^2$ enclosing a simply connected region $D$. Consider the system "NL":

$$\dot{x} = f(x), \quad x \in \mathbb{R}^2$$

Where $f \neq 0$ on $J$. The index of $D$ with respect to $f$ is defined as

$$\text{Idx}(D) = \frac{1}{2\pi} \oint_J \text{d} \Theta_f(x)$$

Where $\Theta_f(x) := \tan^{-1} \frac{f_2}{f_1}$.

Remarks

1. $\Theta_f$ is the angle made by $f$ with the $x_1$-axis:

2. If $\text{Idx}(D)$ is $\frac{1}{2\pi} \times (\text{the net change in the direction of } f \text{ as we traverse } J \text{ counterclockwise})$.
3. If \( f(D) \) is always an integer.

4. If \( x_0 \) is an equilibrium point inside \( D \) and \( D \) encloses no other equilibrium points then \( I_f(D) \) is denoted \( I_f(x_0) \) and is called the index of an equilibrium point.

**Example:**

\[
\begin{align*}
  x_2 & \quad \rightarrow \\
  \rightarrow & \quad x_1
\end{align*}
\]

\[ \Rightarrow I_f(D) = 1. \]

**Example**

\[
\begin{align*}
  x_2 & \quad \rightarrow \\
  \rightarrow & \quad x_1
\end{align*}
\]

Suppose \( I_f(D) = 1 \), can we make a claim as to what is inside \( D \)?
INDEX THEORY - EXAMPLES.

If \( (x_0) \), where \( x_0 \) is:
(a) an unstable node

\[ I_f (x_0) = 1 \]

(b) a stable node

\[ I_f (x_0) = 1 \]

(c) a saddle

\[ I_f (x_0) = -1 \]

(d) a focus

\[ I_f (x_0) = 1 \]

(e) a center

\[ I_f (x_0) = 1 \]

(f)

\[
\begin{align*}
\dot{x}_1 &= x_1^2 - x_2^2 \\
\dot{x}_2 &= 2x_1x_2 \\
x_0 &= (0,0) \\
I_f (x_0) &= 2
\end{align*}
\]
Remark: If $D$ contains finitely many equilibrium points $x_{0i}$, $i=1,2,\ldots,p$, then $\text{If}(D) = \sum_{i=1}^{p} \text{If}(x_{0i})$.

\[ \text{If}(D) = 1 - 1 + 1 = 1 \]

Remark: Let $Y$ be a closed orbit of $f$ enclosing an open set $U$. Then $U$ must contain at least one equilibrium point.

Poincaré Theorem: Let $N$ represent the number of nodes, centers, and foci enclosed by a closed orbit, and let $S$ represent the number of enclosed saddle points. Then $N = S + 1$.

In general, index theory allows you to predict the existence of equilibrium points without doing detailed calculations.
Generalization of Index Theory to higher dimensions:

(called Degree Theory ... chap. 3)

If \( (x_0) = \text{sgn} (\det Df(x_0)) \)

... undefined for \( d = 0 \) ...