GOALS OF THIS LECTURE:

- an introduction to bifurcations through a discussion and classification of bifurcations of equilibrium solutions.

REFS:

SASTRY §2.5, §2.6
KHATIL §2.7.
BIFURCATIONS

Systems of physical interest have parameters which appear in the defining systems of equations:

\[ \begin{align*}
\dot{x}_1 &= f_1(x_1, x_2, \mu) \\
\dot{x}_2 &= f_2(x_1, x_2, \mu)
\end{align*} \iff \dot{x} = f_\mu(x) \quad x \in \mathbb{R}^2 \quad \mu \in \mathbb{R} \]

e.g. Compressor Speed \( B \):

\[ \begin{align*}
\dot{x} &= B(C(x) - y) \\
\dot{y} &= \frac{1}{B}(x - F_{x^{-1}}(y))
\end{align*} \]

\( x^* \) stable focus

\( x^* \) unstable focus

Stable limit cycle

\( B = 0.25 \)

\( B = 0.33 \)
As these parameters are varied, changes occur in the qualitative structure of solutions. We analyzed these by looking at each individual case separately. Now, a systematic study which describes permits analysis of bifurcations.

Equilibrium solutions:

\[ f_\mu(x) = 0 \Rightarrow x^*(\mu), \text{ the equilibrium, is a function of } \mu. \]

\[ x^*(\mu) \text{ is a smooth function of } \mu, \text{ as long as the Jacobian linearization around } x^*(\mu), \text{ denoted } \]
\[ D_x f\mu (x^*(\mu)) = \frac{\partial f\mu}{\partial x} (x^*(\mu)) \]

does not have a zero eigenvalue

- a result of the Implicit Function Theorem.

- What happens when \( D_x f\mu |_{x^*(\mu)} \) has a zero eigenvalue?
  - Several branches of equilibria come together
  - \( (x^*(\mu*)) \) is called a bifurcation point when this occurs.
**Example:**

A scalar \((x \in \mathbb{R}^1)\) system:

\[
\dot{x} = \mu x - x^3, \quad \mu \in \mathbb{R}^1
\]

**Equilibria:**

\[
\mu x^* = x^3 \Rightarrow x^* (\mu - x^{*2}) = 0
\]

\[
x^* = 0
\quad \text{or} \quad
x^* = \pm \sqrt{\mu}
\]

\[
\text{"branches of equilibria"}
\]

\[
Dx f_\mu = \mu - 3x^2
\]

\[
\therefore \quad D x f_\mu / x^* (\mu) \quad \text{has a zero eigenvalue at} \quad (0,0)
\]

---

**Plot of \(x^*\) as a function of \(\mu\):**

- Three branches of equilibria
  - \(x^* = 0\)
  - \(x^* = -\sqrt{\mu}\)
  - \(x^* = +\sqrt{\mu}\)
- Bifurcation point \((0,0)\)
- Superimpose phase portrait
  - \(x^* = \pm \sqrt{\mu}\) stable eq.
  - \(x^* = 0\) stable if \(\mu \leq 0\)
  - \(x^* = 0\) unstable if \(\mu > 0\)
"Pitchfork bifurcation"
EXAMPLE 2:
\[ \dot{x} = \mu x - x^2 \quad x, \mu \in \mathbb{R}^1 \]
equilibria at \( x^* = 0 \) or \( x^* = \mu \)

\[ D_x f(\mu) = \mu - 2x \]

\[ \therefore D_x f(\mu) / x^*(\mu) \] has a zero eigenvalue at \( (0,0) \).

\[ x^* = \mu \} \text{ stable for } \mu > 0 \]
\[ \text{unstable for } \mu < 0 \]

\[ x^* = 0 \} \text{ stable for } \mu < 0 \]
\[ \text{unstable for } \mu > 0 \]

"Transcritical" or "Exchange of Stability"

\( (0,0) \) bifurcation point.
Example 3:

\[ \dot{x} = \mu - x^2 \quad x, \mu \in \mathbb{R}^1 \]

Equilibria at \( x^* = \pm \sqrt{\mu} \)

\( \text{Df}_f \mu = -2x \Rightarrow \) zero eigenvalue at \((0,0)\).

\( x^* = \sqrt{\mu} \) stable

\( x^* = -\sqrt{\mu} \) unstable

\((0,0)\) bifurcation point

"Fold" bifurcation

\[ \text{STRUCTURAL stability of a bifurcation?} \]

\( \dot{x} = \mu - x^2 + \varepsilon \quad \text{FOLD} \)

or

\( \dot{x} = \mu x - x^2 + \varepsilon \quad \text{TRANSCRITICAL} \)

\( \text{does the picture stay qualitatively the same for small perturbations?} \)
Extensions to Planar Systems
\[ x \in \mathbb{R}^2, \quad y \in \mathbb{R}^1 \]

**Example:** Compressor

\[
\begin{align*}
\dot{x} &= B \left( C(x) - y \right) \\
\dot{y} &= \frac{1}{B} \left( x - F_x^{-1}(y) \right)
\end{align*}
\]

Equilibria at \( x^* \) s.t. \( C(x^*) = F_x(x^*) \)

\[ x^* \] is not a function of \( B \) but its stability is:

\[
D_{x} f_B(x^*) = \begin{bmatrix} B \cdot \frac{dc}{dx}(x^*) & -B \\
\frac{1}{B} & -\frac{1}{B \cdot \frac{dF_x}{dx}(x^*)} \end{bmatrix}
\]

We found (HW1) that in the rotating stall region (\( x^* \in (a, b) \))

if \( \frac{dc}{dx}(x^*) < \frac{dF_x}{dx}(x^*) \) then

\[
B < \sqrt{\frac{dc}{dx}(x^*) \cdot \frac{dF_x}{dx}(x^*)} \Rightarrow \text{stable focus}
\]

\[
B > \sqrt{\frac{dc}{dx}(x^*) \cdot \frac{dF_x}{dx}(x^*)} \Rightarrow \text{unstable focus}
\]

\[ \Rightarrow \text{limit cycle?} \]
**Example:**

\[ \begin{align*}
\dot{x}_1 &= -x_2 + x_1 (\mu - x_1^2 - x_2^2) \\
\dot{x}_2 &= x_1 + x_2 (\mu - x_1^2 - x_2^2)
\end{align*} \]

Transform to polar coordinates:

\[ r = (x_1^2 + x_2^2)^{1/2} \]
\[ \theta = \tan^{-1} \frac{x_2}{x_1} \]

\[ \therefore \quad \dot{r} = r (\mu - r^2) \]
\[ \dot{\theta} = 1 \]

Now, original system has equilibrium \((0,0)\) (Aside: equilibria of \((r, \theta)\) system?)

\[ Dxf_{\mu}(x^*) = \begin{bmatrix} \mu & -1 \\ 1 & \mu \end{bmatrix} \]

- Eigenvectors \(\mu + j, \mu - j \Rightarrow \mu < 0 \) stable focus
  \( \mu > 0 \) unstable focus
  \( \mu = 0 \) center

- Also, for \( \mu > 0 \), eq. \((0,0)\) is surrounded by a stable circular limit cycle of
  radius \( \sqrt{\mu} \).
  Check: \( 0 < r < \sqrt{\mu} \Rightarrow \dot{r} > 0 \) \( \text{STABLE} \)
  \( r = \sqrt{\mu} \Rightarrow \dot{r} = 0 \) \( \text{Limit cycle} \)
  \( r > \sqrt{\mu} \Rightarrow \dot{r} < 0 \)
EXAMPLE (cont'd):

\[ x_2 \]
\[ \mu \]
\[ -x_1 \]

"supercritical Hopf Bifurcation"

- stable \( \rightarrow \) unstable, and unstable eq. is surrounded by a stable periodic orbit.

Can also have:

"Subcritical Hopf Bifurcation"

- unstable \( \rightarrow \) stable, and stable is surrounded by an unstable periodic orbit.
**Example:** "Blue Sky Catastrophe"

\[ \begin{align*}
\dot{x}_1 &= -x_1 \sin \mu - x_2 \cos \mu + (1-x_1^2-x_2^2) (x_1 \cos \mu - x_2 \sin \mu) \\
\dot{x}_2 &= x_1 \cos \mu - x_2 \sin \mu + (1-x_1^2-x_2^2) (x_1 \sin \mu + x_2 \cos \mu)
\end{align*} \]

for small \( \mu \).

**Convert to polar coordinates**

\[ \begin{align*}
\dot{r} &= r \left[ (1-r^2)^2 \cos \mu - \sin \mu \right] \\
\dot{\theta} &= (1-r^2)^2 \sin \mu + \cos \mu
\end{align*} \]

For \( \mu < 0 \), one unstable equilibrium pt. at \((0,0)\).

For \( \mu = 0 \), a periodic orbit of radius 1 appears!

**Check:** \[ \dot{r} = r \left[ (1-r^2)^2 \right] \begin{cases} r=1 \Rightarrow \dot{r} = 0 \\ otherwise \Rightarrow \dot{r} > 0 \end{cases} \]

For \( \mu > 0 \), two periodic orbits:

\[ (1-r^2)^2 \cos \mu = \sin \mu \]

\[ \Rightarrow 1-r^2 = \sqrt{\tan \mu} \]

\[ \Rightarrow r = \sqrt{1 \pm \sqrt{\tan \mu}} \]

\[ \Rightarrow \]

- Semi-stable limit cycle
- Unstable limit cycle
- Stable limit cycle

... bizarre behavior