E209A LECTURE 8

Goals of this lecture:

- Define Describing Functions (for SISO nonlinear systems)
- Compute describing functions for single-valued and double-valued skew symmetric functions (i.e., relays, relays with hysteresis)

... Lecture 9: how Describing Functions are used.

Refs:

Sastry 5.4.1 (a more elegant, yet more intense treatment than here)
Khalil 7.2
Describing functions use in NL analysis (assume SISO) sometimes too complicated to fully analyze.

Describing function analysis is an approximation technique, often useful in engineering practice, which may be used to predict the frequency and amplitude of oscillations in systems involving nonlinearities.

\[ e \xrightarrow{\oplus} f(\cdot) \xrightarrow{\oplus} v \xrightarrow{g(s)} y \]

IDEA: determine the "Transfer Function" between the input to \( f \) and the first harmonic (of the input) in the output of \( f \).
\[
\begin{align*}
\text{as} \sin \omega t & \xrightarrow{f(\cdot)} f(\text{as} \sin \omega t) \\
0 \text{ to } T & \xrightarrow{t} T
\end{align*}
\]

\[
f(\text{as} \sin \omega t) = \sum_{k=0}^{\infty} a_k(a) \sin k \omega t + b_k(a) \cos k \omega t
\]

**FOURIER SERIES EXPANSION**

\[
b_0(a) + a_1(a) \sin \omega t + b_1(a) \cos \omega t
\]

where

\[
b_0(a) = \frac{1}{2\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \, dt
\]

\[
a_1(a) = \frac{1}{\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \sin \omega t \, d(\omega t)
\]

\[
b_1(a) = \frac{1}{\pi} \int_{0}^{2\pi} f(\text{as} \sin \omega t) \cos \omega t \, d(\omega t)
\]

- **Approximation made in neglecting** \(K > 1\) terms.
- **Valid approximation if** \(f(\cdot)\) **is part of a larger system which attenuates higher frequencies.**

\[
\sqrt{a_1(a)^2 + b_1(a)^2}
\]

\[
\tan^{-1} \frac{b_1(a)}{a_1(a)}
\]

\[
\approx b_0(a) + \sqrt{a_1(a)^2 + b_1(a)^2} \sin (\omega t + \phi(a))
\]
Define $N(a)$ (the describing function) which is the transfer function between input to $f$ and 1st harmonic in output:

$$N(a) := \frac{\sqrt{a_1(a)^2 + b_1(a)^2}}{a} e^{j \phi(a)}$$

$$= \frac{a_1(a)}{a} + j \frac{b_1(a)}{a}$$

In the above, we made the additional approximation that $b_0(a) = 0$—this is valid for skew-symmetric $f$, i.e.

For $f(x) = -f(-x)$

**SINGLE-VALUED SKEW SYMMETRIC FUNCTION (SVSS)**

which is often a valid approximation since the kinds of nonlinearities often seen in engineering practice are:

RELAYS WITH DEADBAND.

RELAYS WITH HYSTERESIS
EXAMPLES:

I. RELAY WITH DEADBAND.

Clearly, if \( a < \varepsilon \), \( N(a) = 0 \).

Suppose \( a \geq \varepsilon \)

\[
\Rightarrow a_1(a) = 0
\]

\[
\Rightarrow b_1(a) = 0
\]

\[
\Rightarrow a_1(a) = \frac{1}{\pi} \int_0^{2\pi} f(\sin \omega t) \sin \omega t \, d(\omega t)
\]

\[
= \frac{1}{\pi} \cdot 4 \int_0^{\pi/2} \sin \omega t \, d(\omega t)
\]

where \( \sin a = \varepsilon \).
\[ a_1(a) = \frac{4}{\pi} \cos \omega t = \frac{4}{\pi} \sqrt{1 - \frac{E^2}{a^2}} \]

\[ b_1(a) = \frac{1}{\pi} \int_0^{2\pi} f(a \sin \omega t) \cos \omega t \, dt \]

\[ = 0 \quad \text{(all area cancels out over Thus integral) (Fig. **).} \]

\[ \Rightarrow \quad a < E, \quad N(a) = 0 \]

\[ a \geq E, \quad N(a) = \frac{a_1(a)}{a} = \frac{4}{\pi} \sqrt{1 - \frac{E^2}{a^2}} \]

\[ \text{Simplification:} \]

Since \( a_1(a), b_1(a) \) are always independent of \( \omega \): evaluate for \( \omega = 1 \)

\[ a_1(a) = \int_0^{2\pi} f(a \sin t) \sin t \, dt / \pi \]

\[ b_1(a) = \int_0^{2\pi} f(a \sin t) \cos t \, dt / \pi \]
FACT 1 FOR SVM IN f:

\[ b_1(a) = 0 \]

\[ \Rightarrow \mathbf{N}(a) = \frac{a_1(a)}{a} + j \frac{b_1(a)}{a} \in \mathbb{R} \quad \forall a > 0 \]

Why?

\[ b_1(a) = \int_0^{2\pi} f(a \sin t) \cos t \, dt \]

e.g.

\[ f \]

\[ f(a \sin t) \]

\[ \sin t \]

\[ 2\pi \]

\[ t \]

\[ \cos t \]

\[ 2\pi \]

\[ f(a \sin t) \cos t \]

\[ A \]

\[ -A \]

\[ -A \]

\[ \int_0^{2\pi} f(a \sin t) \cos t \, dt = 0. \]
FACT 2 For DVSS function $f$:

**Example**

For $a \geq \bar{a}$, because if $a < \bar{a}$ then the value of $f(\text{asint})$ is not defined when sint decreases after reaching its first maximum.

For $a \geq \bar{a}$:

$$a_1(a) = \frac{1}{\pi} \int_0^{2\pi} \left[ \frac{f_1(\text{asint}) + f_2(\text{asint})}{2} \right] \sin t \, dt$$

$$b_1(a) = \frac{2}{a \pi} \int_0^a \left[ f_1(x) - f_2(x) \right] \, dx$$

$$= \left( -\frac{1}{a \pi} \right) \text{(area of loop)}$$

... makes evaluation of $b_1(a)$ easy
EXAMPLE II. RELAY WITH HYSTERESIS

\[ f \]

\[ f_1 \]

\[ f_2 \]

\[ N(a) \text{ is only sensible for } a \geq \varepsilon \]

**FACT:**

\[ a_1(a) = \frac{1}{\pi^2} \int_{0}^{2\pi} \frac{[f_1(\text{asint}) + f_2(\text{asint})] \text{sint dt}}{2} \]

**FACT 2**

but \( \frac{f_1(\text{asint}) + f_2(\text{asint})}{2} \) is a relay with deadband!

\[ \Rightarrow a_1(a) = \frac{4}{a\pi^2} \sqrt{a^2 - \varepsilon^2} \text{ for } a \geq \varepsilon. \]

**FACT 2**

\[ b_1(a) = \left( \frac{1}{a\pi} \right) \text{ [area of loop]} \]

\[ = -\frac{4\varepsilon}{a\pi} \]

hence, for \( a \geq \varepsilon \),

\[ N(a) = \frac{a_1(a)}{a} + \frac{j b_1(a)}{a} \]

\[ = \frac{4}{a^2\pi} \sqrt{a^2 - \varepsilon^2} - j \frac{4\varepsilon}{a^2\pi} \]
EXAMPLE II (cont'd).

\[ \text{Re.} \]
\[ \text{Im.} \]
\[ N(a) \text{- locus (semi circular)} \]
\[ -\frac{A}{\pi E} \]
\[ a = \varepsilon \]

\[ \text{Re.} \]
\[ \text{Im.} \]
\[ a = \varepsilon \]
\[ -\frac{E\pi}{4} \]
\[ \frac{-1}{N(a)} \text{- locus} \]

EXAMPLE III
\[ f(x) = x^3 \]
\[ f \text{ is SVSS} \Rightarrow b_1(a) = 0 \]
\[ \text{FACT 1} \]

\[ f(asint) = a^3 \sin^3 t = a^3 \left[ 3 \sin t - \sin 3t \right] / 4 \]

since
\[ \sin 3t = 3 \sin t - 4 \sin^3 t \]

keeping only the 1st harmonic:
\[ \frac{3}{4} a^3 \sin t \]
\[ \Rightarrow a_1(a) = \frac{3}{4} a^3 \]

\[ \Rightarrow N(a) = \frac{3}{4} a^2 \]

\[ \frac{-1}{N(a)} \text{- locus} \]

increased \( a \)}