GOALS OF THIS LECTURE:

to show how describing functions may be used in stability analysis of closed loop systems.

REFS

SASTRY G 4.1
KHALIL G 7.2
Using Describing Functions in Stability Studies

**WARNING:** block diagram manipulations for nonlinear systems.

First harmonic terms from each \( f_i \) add to give first harmonic from \( f \).

\[
\Rightarrow N_f(a) = N_{f_1}(a) + N_{f_2}(a) + \cdots + N_{f_i}(a)
\]

**Application:** if we can decompose a given \( f \) as

\[
f(x) = \Xi f_i(x)
\]

for \( f_i \) having known \( N_{f_i} \), then

\[
N_f(a) = \Xi N_{f_i}(a) \quad (\text{easy to find})
\]

However:

**WARNING:** \( N_f(a) \neq N_{f_1}(a) \cdot N_{f_2}(a) \) \& in general

and similarly for several functions \( f_i \) in series.
Also: for linear systems.

\[
\begin{array}{c}
\rightarrow g_1(s) \rightarrow g(s) \rightarrow \Xi \\
\rightarrow g_2(s) \rightarrow g(s) \\
\end{array}
\]

\[\equiv\]

\[
\begin{array}{c}
\rightarrow g_1 \\
\rightarrow g_2 \\
\end{array}
\]

but for non-linear systems

\[
\begin{array}{c}
\rightarrow g_1(s) \rightarrow n \rightarrow \Xi \\
\rightarrow g_2(s) \rightarrow n \\
\end{array}
\]

\[\not\equiv\]

because, for non-linear \( n \), superposition is not valid.

For non-linear systems, manipulations are OK iff inputs to non-linearities remain unaltered.

\[
\begin{array}{c}
\rightarrow \Xi \\
\rightarrow g_1 \rightarrow n \rightarrow g_2 \\
\end{array}
\]

\[\equiv\]

\[
\begin{array}{c}
\rightarrow g_1 \rightarrow \Xi \\
\rightarrow n \\
\rightarrow \left( g_2 + g_1, g_3 \right) \\
\end{array}
\]
Many nonlinear systems can be written as:

\[ O \xrightarrow{+} e \xrightarrow{\leq} n \xrightarrow{v} g(s) \xrightarrow{} y \]

where \( n \) is a SVSS or DVSS function.

Concentrate on error \( e \): since if we know what happens to \( e \), we can determine what happens elsewhere in the system, easily.

- Will it oscillate?
- What amplitude?
- What \( \omega \)?

**Assume**

\[ e \xrightarrow{=} \text{sustained sinusoidal oscillation} \]

\[ \Rightarrow e(t) = a \sin(\omega t + \theta) \text{ for some } a, \omega, \theta \]

**Then**

\[ v(t) = n(e(t)) \]

\[ = n\left(a \sin(\omega t + \theta)\right) \]

\[ = a |N(a)| \sin(\omega t + \theta + \phi(a)) + \text{higher order terms} \]

where \( N(a) \) is the transfer function between the input to \( n \) and the 1st harmonic in the output of \( n \), and \( \phi(a) \) is the phase angle associated with \( N(a) \).
\[ e(t) = a \sin (wt + \theta) \]
\[ v(t) = a \left| N(a) \right| \sin (wt + \theta + \phi(a)) + \text{higher harmonics.} \]

**Assume:** \( g(s) \) attenuates higher harmonics, in that \( g \) looks like: \(|g|\)

Then \( y(t) \approx \) result of \( g \) operating on 1st harmonic in \( v \)
\[ \approx |g(jw)| a \left| N(a) \right| \sin (wt + \theta + \phi(a) + \gamma(w)) \]

So \[ e(t) = -y(t) \]
\[ \Rightarrow a \sin (wt + \theta) \approx -|g(jw)| a \left| N(a) \right| \sin (wt + \theta + \phi(a) + \gamma(w)) \]

**Assume this is actually**
\[ a \sin (wt + \theta) = -|g(jw)| a \left| N(a) \right| \sin (wt + \theta + \phi(a) + \gamma(w)) \]

In phasor form:
\[ a e^{j(wt + \theta)} = -|g(jw)| a \left| N(a) \right| e^{j(wt + \theta + \phi(a) + \gamma(w))} \]
\[ = -a \left| g(jw) \right| e^{j\gamma(w)} \left| N(a) \right| e^{j\phi(a)} e^{j(wt + \theta)} \]
i.e. $1 = -\frac{g(jw)}{e^{j\gamma(w)}} \cdot \frac{|N(a)|e^{j\varphi(a)}}{g(jw)}$

Harmonic balance equation

$g(jw) = -\frac{1}{N(a)}$

Relationship between $w$ and $a$

Assumptions made:
- $e(t) = a \sin (wt + \theta)$
- $g$ attenuates higher frequencies
- Describing function is exact

Example:

Assume: $g(s) = \frac{40}{s(s^2+2)(s+8)}$

Suggests: if intersection takes place, then oscillations may occur in $e$ with the $w$, $a$ associated with the intersection point.
But if we had: 

\[ g(jw) = -\frac{1}{N(a)} \]

not true for any \( w, a \),

which suggests no oscillation (because if \( e(t) = a \sin(\omega t + \theta) \) and assumptions all valid, then \( g(jw) = -\frac{1}{N(a)} \) must be true for some \( w, a \).

**Note:** Predictions not foolproof in that, owing to the approximations involved in the analysis:

- predicted oscillations might not happen
- predicted no-oscillations might be false

Nonetheless, this is often a useful tool.

Q: Can we predict whether oscillations will decay, be sustained, or explode?

A: Yes... using an extension of Nyquist.
Nyquist:

\[ \begin{array}{c}
\text{constant gain} \\
\leftarrow \\
K \\
\rightarrow \\
\text{g(s)} \\
\end{array} \]

Then: closed loop poles are in open left half plane iff complete \( g(j\omega) \) - locus encircles \( \left[ -\frac{1}{K} + j0 \right] \) \( \times \) times (anticlockwise)

\[ \text{# OL poles in open RHP} \]

eg. \( p = 0 \)

- \( \frac{1}{K} \) here: all poles in open LHP.

\[ \text{\( \frac{1}{K} \) here: at least one closed loop pole in closed rhp} \]

Actually, true for all \( k \in \mathbb{C} \):

eg. \( p = 0 \)

- \( \frac{1}{k} \) here: all poles in gain rhp

\[ \text{-\( \frac{1}{k} \) here: at least one closed loop pole in } \]

eg. \( p = 0 \)

\[ \text{closed rhp.} \]

\[ \text{at least one closed loop pole in closed rhp if } -\frac{1}{k} \in \mathbb{R} \]
Application to:

\[ 0 \xrightarrow{\leq} e \xrightarrow{n} g(s) \rightarrow y \]

For \( e(t) = a \sin(\omega t + \theta) \) view \( g \) as approximated by:

\[ k = N(a) \in \mathbb{C} \]

because the complex number \( N(a) \) is the transfer function between the input sinusoid to \( n \) and the first harmonic in the output of \( n \).

Also, assume \( N(a) \) works for \( e(t) \) of the form \( ae^{\lambda t} \sin(\omega t + \theta) \) [as well as the case we just did - for \( e(t) \) of the form \( a \sin(\omega t + \theta) \)]

Then, growth or decay of \( e(t) \) is predictable from position of \( -\frac{1}{k} = -\frac{1}{N(a)} \) with respect to \( g(j\omega) \)-locus.

\[ \text{eg: if } -\frac{1}{N(a)} \text{ is here} \]

\[ \text{growing } e(t) \] (if \( -\frac{1}{N(a)} \) is here)
Hence can predict \( e(t) \) behavior:

- Say \( e(0) = 0 \) and \( p = 0 \)

\[ \begin{align*}
\text{start here as } e(0) &= 0; \text{ corresponds to } a = 0 \\
\text{initially oscillation grows as } &-\frac{1}{N(a)} e \rightarrow \text{III} \\
\Rightarrow & a \text{ increased} \\
\Rightarrow & -\frac{1}{N(a)} \text{ move to b, but stops at b as} \\
\text{e(t)} & \text{ and hence a decreased on left of b} \\
\Rightarrow & \text{sustained oscillation with a, w values those at b.} \\
\Rightarrow & \text{means "implies, more or less"}
\end{align*} \]

# Since:

\[ -\frac{1}{N(a)} e \rightarrow \text{growing amplitude a of assumed sinusoid} \]

\[ -\frac{1}{N(a)} e \text{ unshaded } \rightarrow \text{decaying amplitude a of assumed sinusoid e.} \]
WARNING:

Describing function analysis based on many approximations ➞ predictions not necessarily correct

Can redo theory for:

$$e(t) = \sum_{i=1}^{r} a_i \sin (w_i t + \theta_i)$$

$$\downarrow$$

Multiple-input describing functions

$$\downarrow$$

For which, can be shown: If a closed loop oscillation exists, it can be predicted for some finite r,

and for which:

error bounds can be obtained.