1.) **Phase-locked loop**

\[ y(t) + (a + b \cos(y(t)))\dot{y} + c \cdot \sin(y(t)) = 0 \]

Let \( x_1 = y, \ x_2 = \dot{y} \)

\[ \Rightarrow \dot{x}_1 = x_2 \]

\[ \dot{x}_2 = -(a + b \cos x_1)x_2 + c \sin x_1 \]

\( x_e = (0, 0) \) is an equilibrium

**Lyapunov function candidate**

\[ V(x_1, x_2) = C \left( x_1 \right) \]

\[ = C \left( \sin \frac{x_1}{2} \right) \]

Let \( G_r = \{ x \in \mathbb{R}^2, \ l\|x\| \leq \pi \} \), \( \Rightarrow \|x_1\| < \pi \)

From the plot above, we can see

\[ \left| \sin \frac{x_1}{2} \right| \leq \frac{x_1}{2} \leq \frac{\pi}{2} \]

on \( G_r \)

\[ \Rightarrow 2C \left( \frac{x_1}{2} \right)^2 + \frac{x_2^2}{2} \leq V(x_1, x_2) \leq 2C \left( \frac{x_1}{2} \right)^2 + \frac{x_2^2}{2} \]

\[ \Rightarrow \min \left\{ \frac{2C}{\pi^2} \cdot \frac{\pi^2}{4}, \frac{1}{2} \right\} \|x\|^2 \leq V(x_1, x_2) \leq \max \left\{ \frac{C}{2}, \frac{1}{2} \right\} \|x\|^2 \]

\[ \Rightarrow V \text{ is PD and decreasing on } G_r \]

\( \square \)
\[ \dot{V} = c \sin x_1 \dot{x}_1 + x_2 \dot{x}_2 \]
\[ = c \sin x_1 \dot{x}_2 + x_2 \left( -c \sin x_1 - (a + b \cos x_1) x_2 \right) \]
\[ = -(a + b \cos x_1) x_2^2 \]

\((a)\) if \(a \geq b \geq 0\)
\[ a + b \cos x_1 \geq b (1 + \cos x_1) \geq 0 \]
\[ \Rightarrow \dot{V} \leq 0 \]

\(V\) is a Lyapunov function. ②

From ① and ②
\[ \Rightarrow x_e = (0, 0) \text{ is stable.} \]

\((b)\) if \(a > b > 0\), use LaSalle's Theorem.
\[ \mathcal{L}_c = \{ x \in \mathbb{R}^2, \ V(x) \leq c \} \]

from (a), \(\mathcal{L}_c\) is bounded and \(\dot{V}(x, t) \leq 0\).
\[ S = \{ x \in \mathcal{L}_c, \ V = 0 \} \]
\[ = \{ x \in \mathcal{L}_c, \ -(a + b \cos x_1) x_2^2 = 0 \} \]
\[ \Rightarrow a > b, \ a + b \cos x_1 > 0 \]
\[ S = \{ x \in \mathcal{L}_c, \ x_2 = 0 \} \]

on \(S\), \( x_2 = 0, \ \dot{x}_2 = 0 \Rightarrow \sin x_1 = 0 \)
\[ \Rightarrow x_1 = 0 \quad (\text{assume } \mathcal{L}_c \subset \mathbb{R}^2, \text{ hence exclude } x_1 = k \pi) \]
\[ \Rightarrow x_e = (0, 0) \text{ is asymptotically stable.} \]
2.) \[
\begin{align*}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + (1-x_1^2-x_2^2)x_2
\end{align*}
\]

(a) \[
\begin{bmatrix}
0 \\
1
\end{bmatrix}
\begin{bmatrix}
0 & 1 \\
-1 & 1
\end{bmatrix}
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\]

\[
\lambda = 0.5 \pm 0.866 i ; \quad \text{Unstable Focus}
\]

(b) We suspect limit cycle at \( x=1 \)

\[
\begin{align*}
&\text{Let } x^2 = x_1^2 + x_2^2 \\
&\Rightarrow 2x^2 \ddot{x}_1 = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 \\
&\Rightarrow \dot{x}_1 = x_2^2 (1-x_1^2-x_2^2) \\
&\Rightarrow \dot{x}_1 = x_2^2 (1-x_1^2)/\lambda
\end{align*}
\]

\[
\Rightarrow \text{When } x=1
\]

\[
\dot{x} = 0 \quad \Rightarrow \text{Limit cycle at } x=1
\]

\[
\dot{\theta} = -1 
\]

(c) Choose cake-pan shaped Lyapunov Function.

\[
V(x) = \frac{1}{x^2} + x^2 - 2
\]

\[
\Rightarrow \dot{V}(x) = \frac{1}{x^2 + x^2} x^2 + x_1 \dot{x}_2 - 2
\]

\[
\dot{V}(x) = (2\dot{x} - \frac{2}{x^2})\dot{x} = (2\dot{x} - \frac{2}{x^2})x^2 (1-x^2)/\lambda
\]

Prop. Jacobi’s Principle:

\[
\begin{align*}
&\text{For } x \in \mathbb{R}^2 \\
&2x^2 (x^2 - 1) (1-x^2) \leq 0 \quad \text{for } x \in \mathbb{R}^2
\end{align*}
\]

(Check this!)

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\[
\Rightarrow \dot{V}(x) \leq 0 \\
\Rightarrow \{ x : \dot{V}(x) = 0 \}
\]

\[
\Rightarrow S = \{ x : x_1 = 0 \text{ or } x_1 = 1 \}
\]

Largest Invariant Subset of \( S \).

\[
\text{If } x_2 = 0 \text{ and } x_1 \neq 0, 1 \Rightarrow x_2 = 0 \text{ because } S.
\]

\[
\Rightarrow M = \{ x : x_1 = 1 \} \text{ on } x_1^2 + x_2^2 = 1 \quad \Rightarrow \text{All trajectories converge to } M.
\]

Not from (0,0) which is the limit cycle.
Satellite Stabilization.

3.)

- States: \([w, \dot{w}] \in \mathbb{R}^6\)

- System equations:

\[
\begin{align*}
I\ddot{w} + w \times Iw &= u \\
\dot{d}_0 &= -w \times d_0 \\
\text{with } u &= -d_\omega + d_0 \times b_0, \quad d_\omega > 0
\end{align*}
\]

- Find equilibria:

\[
\dot{w} = 0 \quad \text{and} \quad \dot{d}_0 = 0
\]

\[
\Rightarrow \begin{cases}
wxIw = u = -d_\omega + d_0 \times b_0 & \cdots \, \circ \\
wx d_0 &= 0 & \cdots \cdots \, \circ
\end{cases}
\]

From \(\circ\), \(\text{suppose } w \neq 0\)

\[
d_0 = k_1w, \quad k_1 \text{ is a constant scalar.}
\]

Plug into \(\circ\),

\[
w \times Iw - k_1(w \times b_0) = -d_\omega
\]

the left hand side of the above equation is perpendicular to \(w\),

the right hand side is in the direction of \(w\).

\[
\Rightarrow \quad w = 0
\]

Now from \(\circ\),

\[
d_0 \times b_0 = 0 \quad \Rightarrow \quad d_0 = k_2b_0, \quad k_2 \text{ is a constant.}
\]

Since \(d_0\) and \(b_0\) are unit vectors

\[
\Rightarrow \quad d_0 = \pm b_0 \quad (\text{lined up})
\]

equilibria:

\[
\begin{bmatrix}
0 \\
0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
0 \\
b_0
\end{bmatrix}
\text{ and }
\begin{bmatrix}
0 \\
-b_0
\end{bmatrix}
\]
we now use Lyapunov theory to determine the stability of the equilibria.

\[ V(w, do) = \frac{1}{2} \dot{w}^T \dot{I} w + \frac{1}{2} \| \dot{d}o + \dot{b}o \|^2 \]

\[ V(o, bo) = 2 \]

\[ V(o, -bo) = 0 \]

Since \( I > 0 \), \( V(w, do) \) is PD.

\[ \dot{V} = \frac{1}{2} \dot{w}^T \dot{I} w + \frac{1}{2} \dot{w}^T \dot{I} \dot{w} + \frac{1}{2} (\dot{d}o + \dot{b}o)^T (\dot{d}o + \dot{b}o) + \frac{1}{2} (\dot{d}o + \dot{b}o)^T (\dot{d}o + \dot{b}o) \]

\[ = \dot{w}^T \dot{I} w + (\dot{d}o + \dot{b}o)^T (\dot{d}o + \dot{b}o) \text{ since } \dot{w}^T \dot{I} w = w^T \dot{I} \dot{w} \]

\[ \dot{b}o = 0 \text{ because in the body frame, } \dot{b}o \text{ is fixed.} \]

\[ \Rightarrow \dot{V} = \underline{\dot{w}^T \dot{I} w + \dot{d}o^T (\dot{d}o + \dot{b}o)} \text{ (note, the underlined are equal & } I^T = I) \]

\[ = (\underline{-w^T w + w^T \dot{x} b_0} - w^T w) + (-w^T d o) + (w^T d o) \]

\[ = -\dot{w}^T w + (w^T d o)^T \dot{w} + (-w^T w)^T \dot{w} - (w^T d o)^T d o - (w^T d o)^T b_0 \]

Using \( F \cdot G \times H = G \cdot H \times F = H \cdot F \times G \)

\[ \therefore (w^T d o)^T \dot{w} = (w^T d o)^T b_0 \]

\[ (-w^T w)^T \dot{w} = (w^T d o)^T d o = 0 \]

\[ \Rightarrow \dot{V} = -\dot{w}^T \dot{w} = -\dot{w} \| \dot{w} \|^2 \leq 0 \]

System is SISL.

Applying Lasalle's theorem:

Looking at \( xe = \begin{bmatrix} 0 \\ -bo \end{bmatrix} \).
Choose $G < 2$, and define

$$\mathcal{N}_C := \{ (w, do) \in \mathbb{R}^b, \|d\| = 1 \mid V(w, do) \leq G \}$$

So $[0, 0] \notin \mathcal{N}_C$.

Define $S \subset \mathcal{N}_C$ as

$$S := \{ (w, do) \in \mathcal{N}_C \mid \dot{V}(w, do) = 0 \}$$

$$= \{ (w, do) \in \mathcal{N}_C \mid \|d\| \|w\|^2 = 0 \}$$

$$= \{ (w, do) \in \mathcal{N}_C \mid w = 0 \}$$

The largest invariant set in $S$ is

$$M := \{ (w, do) \in S \mid w = 0 \text{ and } \dot{w} = 0 \}$$

on $M$, $w = 0 \Rightarrow \dot{w} = -w \times do = 0$ (equilibria)

since

$$M \subset S \subset \mathcal{N}_C \text{ and } [0, 0] \notin \mathcal{N}_C$$

$\Rightarrow M = \{(0, -bo)\}$

By LaSalle's Theorem,

$(0, -bo)$ is locally asymptotically stable.

$\Rightarrow$ all initial conditions in $\mathcal{N}_C$ converge to $(0, -bo)$. 

4.) \( \text{Try } V(w) = \frac{1}{2} \left( J_1 w_1^2 + J_2 w_2^2 + J_3 w_3^2 \right) \)

(a) Updf: 

\[ V(w) = J_1 w_1 w_1 + J_2 w_2 w_2 + J_3 w_3 w_3 \]

\[ \dot{V}(w) = 0 \]

\( \Rightarrow \) origin is stable 

it is not asymptotically stable - show!

(ii) \( u_i = -k_i w_i \)

using same \( V(w) \), we have 

\[ \dot{V} = -k_1 w_1^2 - k_2 w_2^2 - k_3 w_3^2 < 0 \]

\( \therefore \) origin is globally asymptotically stable.

5.)

(i) \( \dot{x}_1 = x_2, \quad \dot{x}_2 = -g(x_1) - h(x_1) x_2 \)

@ equilibrium,

\[ x_2 = 0 \quad \Rightarrow \quad g(x_1) + h(x_1) x_2 = 0 \]

\[ \Rightarrow \quad x_2 = 0 \quad \Rightarrow \quad g(x_1) = 0 \]

if we assume that \( g(x, \theta) = 0 \) has an isolated root at the origin, then the origin is an isolated equilibrium point.

(ii) Take candidate Lyapunov fn as, in class, 

\[ V(x) = \int_{0}^{x_1} g(y) dy + \frac{1}{2} x_2^2 \]

\[ V(x) = g(x_1) x_2 - x_2 g(x_1) + h(x_1) x_2^2 \]

\[ \Rightarrow \quad -h(x_1) x_2^2 \leq 0 \]
now, if \( h(x) > 0 \) \( \forall x \in D \) (D contains origin)

\[ y' = 0 \iff h(x) x^2 = 0 \Rightarrow x^2 = 0 \]

\[ 0 = h(x) \iff x = 0 \]

hence, by LaSalle, origin is asymptotically stable.
Problem 6

The governing equations are:

\[ \dot{x}_1 = x_2 \]
\[ \dot{x}_2 = x_1 - n(2x_1 + x_2) \]

where

\[ n(y) = \begin{cases} 
-1 & y < -1 \\
\quad y & |y| \leq 1 \\
1 & y > 1 
\end{cases} \]

Let’s try the Lyapunov candidate \( V = \frac{1}{2}||x||^2 \) which is by definition positive definite and decrescent. Taking the Lie derivative of \( V \) yields:

\[ \dot{V}(x) = 2x_1x_2 - x_2n(2x_1 + x_2) \]

Now let’s try to find a region in which we can guarantee that \( V(x) \leq 0 \). If \( |2x_1 + x_2| \leq 1 \), then

\[ \dot{V}(x) = -x_2^2 \leq 0 \]

Therefore, we can at least say that the equilibrium point is stable in the sense of Lyapunov, but since \( \dot{V}(x) \) is not positive definite, let’s try to use Lasalle’s theorem to prove local asymptotic stability. First we need to define the \( \Omega_c \) region in which \( \dot{V}(x) \leq 0 \).

\[ |2x_1 + x_2| \leq 1 \]
\[ |2x_1 + x_2| \leq 2||x|| + ||x|| = 3||x|| \leq 1 \]

Writing this in terms of our Lyapunov candidate we obtain

\[ \Omega_c = \left\{ x \mid V \leq \frac{2}{9} \right\} \]

The subset of \( \Omega_c \) in which \( \dot{V}(x) = 0 \) is

\[ S = \left\{ x \mid \dot{V}(x) = 0 \right\} \]

which is \( x_2 = 0 \). The invariant subset of \( S \) is

\[ x_1 - n(2x_1 + x_2) = 0 \]
\[ x_1 - n(2x_1) = 0 \]

which can only be zero at \( x_1 = 0 \). Thus, the invariant subset of \( S \) is

\[ M = \left\{ x \mid x_1 = 0, x_2 = 0 \right\} \]

Therefore, by Lasalle’s theorem, the origin is locally asymptotically stable.