\( \dot{x}_1 = k_1 x_2 \left( 1 - \frac{x_1}{1 + x_2^2} \right) \)
\( \dot{x}_2 = k_2 - x_2 - \frac{4x_1x_2}{1 + x_2^2} \)
\( k_1, k_2 > 0 \)

(a) **equilibria:**
\[
-k_1 x_2 \left( 1 - \frac{x_1}{1 + x_2^2} \right) = 0 \quad (1)
\]
\[
k_2 - x_2 - \frac{4x_1x_2}{1 + x_2^2} = 0 \quad (2)
\]

From (1), \( x_2 = 0 \) or \( \frac{x_1}{1 + x_2^2} = 1 \), but \( x_2 = 0 \Rightarrow k_2 = 0 \) \( \Rightarrow x_2 \neq 0 \) \( \Rightarrow \) unique equilibrium @ \( (1 + \left( \frac{k_2}{5} \right)^2, \frac{k_2}{5}) \)

\[
\text{Linearise: } \frac{\partial f}{\partial x} = \begin{bmatrix}
-k_1 x_2 \\
-4x_2
\end{bmatrix} - \begin{bmatrix}
\frac{k_1 x_1}{1 + x_2^2} + \frac{2k_1 x_1 x_2}{1 + x_2^2} \\
\frac{4x_1}{1 + x_2^2} + \frac{8x_1 x_2^2}{1 + x_2^2}
\end{bmatrix}
\]

@ equilibrium, \( \frac{x_1}{1 + x_2^2} = 1 \)

\[
\left. \frac{\partial f}{\partial x} \right|_{eq} = \frac{1}{1 + x_2^2} \begin{bmatrix}
-k_1 x_2 & 2k_1 x_2^2 \\
-4x_2 & -5 + 3x_2^2
\end{bmatrix}
\]

:. characteristic equation:
\( s^2 + s(5 - 3x_2^2 + k_1 x_2) + 5k_1 x_2 = 0 \)

@ equilibrium, \( s^2 + s(5 - 3\left( \frac{k_2}{5} \right)^2 + \frac{k_1 k_2}{5}) + k_1 k_2 = 0 \) (x)
(b) Consider the region
\[ K = \{ (x_1, x_2) \mid x_1 > 0, x_2 > 0, x_1 \leq 1 + k_2^2, x_2 \leq k_2^2 \} \]

It is easy to show invariance of \( K \):
- on \( x_1 = 0 \), \( \dot{x}_1 = k_1 x_2 > 0 \) if \( x_2 > 0 \)
- on \( x_2 = 0 \), \( \dot{x}_2 = k_2 > 0 \)
- on \( x_1 = 1 + k_2^2 \), \( \dot{x}_1 = k_1 x_2 (1 - \frac{1 + k_2^2}{1 + x_2^2}) < 0 \) if \( 0 < x_2 < k_2 \)
- on \( x_2 = k_2 \), \( \dot{x}_2 = -4x_1k_2 \frac{k_2}{1 + k_2^2} < 0 \) for \( x_1 > 0 \)

\[ \therefore K \text{ is invariant} \]

\[ \therefore \text{in order to have (by Poincaré–Bendixson) a closed orbit in } K, \text{ we need that the only equilibrium inside } K \text{ be unstable.} \]

(Note that the equilibrium \( @ (1 + (\frac{k_2}{5})^2, \frac{k_2}{5}) \) is inside \( K \).

Examining the characteristic eqn. (*), since \( k_1, k_2 > 0 \), a necessary sufficient condition for the equilibrium to be unstable is that \( 5 - 3 \left( \frac{k_2}{5} \right)^2 + \frac{k_1k_2}{5} < 0 \).

\[ \therefore \text{By Poincaré–Bendixson, the system is guaranteed to have a periodic orbit in } K \text{ if} \]
\[ 5 - 3 \left( \frac{k_2}{5} \right)^2 + \frac{k_1k_2}{5} < 0 \]

or \( k_1 < \frac{3k_2}{5} - \frac{25}{k_2} \)

(c) When \( k_1 > \frac{3k_2}{5} - \frac{25}{k_2} \), equilibrium \( k_2 \) is stable
when \( k_1 < \frac{3k_2}{5} - \frac{25}{k_2} \), equilibrium is unstable
is surrounded by an orbit

\[ \Rightarrow \text{Supercritical Hopf bifurcation} \text{ at } k_1 = \frac{3k_2}{5} - \frac{25}{k_2} \]
2 (a) \[
\begin{align*}
\dot{x}_1 &= 1 - x_1 x_2^2 \\
\dot{x}_2 &= x_1
\end{align*}
\]

(i) \text{div}(f) = -x_2^2 \\
(ii) \text{equilibria do not exist} \ldots

This system has no orbits, because, by Index Theory, an orbit must surround an equilibrium point. Thus, no equilibria \Rightarrow no closed orbits.

(b) \[
\begin{align*}
\dot{x}_1 &= x_2 \cos x_1 \\
\dot{x}_2 &= \sin x_1
\end{align*}
\]

\text{(Soln 1)} \\
(i) \text{simple solution: note that all equilibria are saddles. Thus, by Index Theory, no closed orbits exist!}

\text{now } \text{div}(f) \equiv 0 \text{ if } x_2 = 0 \text{ or } x_1 = n\pi.

\text{if we partition the state space into regions defined by } x_2 = 0 \text{ and } x_1 = n\pi, \text{ we know that, from Bendixson's Thm, that an orbit cannot be completely enclosed in one of these regions:}

\text{regions } A \text{ are positively invariant, regions } B \text{ are negatively invariant.}

\text{there are no closed orbits in this system, since if there were, they would have to leave a region } A \text{ and enter a region } B, \text{ which is impossible.}
3. (a) The solution of the state equation

\[ \begin{align*}
\dot{x}_1 &= x_2, \quad x_1(0) = x_{10} \\
\dot{x}_2 &= k, \quad x_2(0) = x_{20}
\end{align*} \]

where \( k = \pm 1 \), is given by

\[ \begin{align*}
x_2(t) &= k t + x_{20} \\
x_1(t) &= \frac{1}{2} k t^2 + x_{20} t + x_{10}
\end{align*} \]

Eliminating \( t \), we obtain:

\[ x_1 = \frac{1}{2k} x_2^2 + c \quad (c = x_{10} - \frac{x_{20}^2}{2k}) \]

Plotted below for different \( c \):

![Plotted graphs](image)

From the superimposed plot below, we see that trajectories can reach the origin only through 2 curves (highlighted).

Switching policy:

1. If start at pt to the right (i.e. A) apply \( u = -1 \) until hit curve, then switch to \( u = 1 \)
2. B, apply \( u = 1 \) until hit curve, then apply \( u = -1 \)

Important: at origin? (apply \( u = 0 \))
3(b) \( G(s) = \frac{1}{s^2} = \frac{\Theta(s)}{u(s)} \)

Plant equation:
\[
\dot{\theta} = u \\
\epsilon(t) = r(t) - \Theta(t) \\
\dot{\epsilon}(t) = -\dot{\theta}(t) \\
\ddot{\epsilon}(t) = -\ddot{\theta}(t)
\]

\[\therefore \quad -\ddot{\epsilon} = n(\epsilon) \quad \Rightarrow \quad \frac{d\dot{\epsilon}}{d\epsilon} = -\frac{n(\epsilon)}{\ddot{\epsilon}}
\]

Three regions:
1. \( \epsilon < -0.2 \Rightarrow \frac{d\dot{\epsilon}}{d\epsilon} = \frac{1}{\epsilon} \)
2. \(-0.2 < \epsilon < 0.2 \Rightarrow \frac{d\dot{\epsilon}}{d\epsilon} = 0 \)
3. \( \epsilon \geq 0.2 \Rightarrow \frac{d\dot{\epsilon}}{d\epsilon} = -\frac{1}{\epsilon} \)

\( \epsilon(0) = 0.2 \Rightarrow \Theta(0) = 0.28 \)
\( \dot{\epsilon}(0) = 2 \Rightarrow \ddot{\epsilon}(0) = -2 \)
Claim: \( e(t) = \alpha + \beta \dot{e}(t)^2 \) for some \( \alpha, \beta \)
\( \dot{e}(t) = 2\beta \dot{e}(t) \ddot{e}(t) \).

\[ n(e) = 1 \Rightarrow \dot{e} = \frac{1}{2\beta} \text{ if } \dot{e}(t) \neq 0. \]

Also, if \( \dot{e} = 0 \) then \( \beta = \frac{1}{2} \alpha. \)
Also, \( e(0) = \alpha - \frac{1}{2} \dot{e}(0)^2 \Rightarrow \alpha = 2.2 \)
\( e(t) = 2.2 - \frac{1}{2} \dot{e}(t)^2 \)

Similarly, for \( n(e) = -1 \), \( e(t) = -1.8 + \frac{1}{2} \dot{e}(t)^2 \)
\( e(t) = 2.2 - \frac{1}{2} \dot{e}(t)^2 \)
\[ \Rightarrow \dot{e}(t) = \sqrt{4.4 - 2e(t)} \]
\[ \Rightarrow \frac{de}{dt} = \frac{\sqrt{4.4 - 2e}}{\sqrt{4.4 - 2e}} \text{ for } \dot{e} > 0 \]
\[ \Rightarrow -\frac{2}{\sqrt{4.4 - 2e}} \bigg|_{0.2}^{2.2} = t \]
\[ \Rightarrow t = -(0.2 - 2) = 2 \text{ seconds}. \]

Time to go from \((-0.2, 2)\) to \((0.2, 2)\)
\[ \ddot{e} = 0 \]
\[ \therefore \dot{e} = k = 2 \]
\[ \Delta e = kt \Rightarrow e = 2 \cdot t \]
\[ \Rightarrow \text{Time for one period is: } 4 \times 2 + 2 \cdot 0.2 = 8.4 \text{ seconds} \]
for $e > 0.2$

$\dot{e} = -n(e) = -1$  \hspace{1cm} \text{(constant)}

at A, $\dot{e} = 2$

at B, $\dot{e} = 0$

from $A \rightarrow B$

$\tau_{AB} = \frac{\dot{e}B - \dot{e}A}{\dot{e}} = \frac{0.2 - (-0.2)}{-1} = 2 \text{ sec}$

from $F \rightarrow A$

$\dot{e} = 2$  \hspace{1cm} \text{(constant)}

$\tau_{FA} = \frac{\dot{e}A - \dot{e}F}{\dot{e}} = \frac{0.2 - (-0.2)}{2} = 0.2 \text{ sec}$

symmetric, so period

$T = 4 \cdot \tau_{AB} + 2 \cdot \tau_{FA} = 4 \times 2 + 2 \times 0.2 = 8.4 \text{ sec}$
\[ T = -T + \int f(T) \]

\[ T(t) = e^{-t} \cdot T(t_{on}) + \int_{t_{on}}^{t} e^{-(t-t')} \cdot 100 \, dt' \]

\[ = T(t_{on}) e^{-t} + (1 - e^{t_{on}-t}) \cdot 100 \]

\[ \uparrow \text{Time @ which heater is switched on} \]

Heater off:

\[ T(t) = T(t_{off}) e^{-t} \]

\[ \uparrow \text{Time @ which heater is switched off} \]

\[ T(t) \]

\[ 60 \]

\[ 50 \]

\[ 40 \]

\[ \text{oscillated with frequency and ampitude independent of } T(0). \]
\[ \frac{1}{s+1} f(T) = T \]

b. \( e = 50 - T \)

\[ \dot{e} = -T \quad \text{and thus} \quad -\dot{e} + 50 - e = f(50 - e) \]
\[ -(\dot{e} + e) = -50 + f(50 - e) \]
\[ \dot{e} + e = 50 - f(50 - e) \]

\[ \Rightarrow \dot{e} + e = n(-e) \]

where \( n(-e) = 50 - f(50 - e) \)

Note that:

- Diagram showing the function and its derivatives with arrows indicating the flow.

Which is what we want!
Thus we rewrite the system as:

\[ 0 + \text{n(e)} - \frac{1}{s+1} \text{e} \]

\[ \mathbf{e} + \mathbf{e} = \text{n(e)} \]

We know from class that for the hysteresis element, the DF (for \( a = \Delta \)) is

\[ N(a) = \left( \frac{4}{\pi} \sqrt{a^2 - \Delta^2} - j \frac{4\Delta}{\pi} \right) \cdot 50 \]

from HW 5 problem.

\[ \therefore \frac{-1}{N(a)} = \frac{a^2 \pi}{200} \frac{1}{\sqrt{a^2 - \Delta^2} - j \Delta} = \frac{\sqrt{a^2 - \Delta^2} + j \Delta}{\sqrt{a^2 - \Delta^2} + j \Delta} \]

\[ = -\frac{\pi}{200} \left( \frac{a^2}{\sqrt{a^2 - \Delta^2} + j \Delta} \right) \]

doesn't depend on \( a \)!

\[ \Rightarrow \text{predicted no oscillation!} \]

\[ \therefore \text{DF prediction is wrong!} \]

\[ \frac{-1}{N(a)} \]

\[ \Delta \]

\[ s+1 \]