1. (a) Since there is a positively invariant closed set containing the equilibrium at 

\[(u_e = a + b, \ v_e = \frac{b}{(a+b)^2})\], Thus, in order to show that a closed orbit exists, we show that this set excluding the equilibrium is positively invariant, by choosing \(a, b\) so that the equilibrium is an unstable node or focus. Thus, by Poincaré–Bendixson, we have the result.

\[
Jacobian \ J = \begin{bmatrix}
-1 + 2uv & u^2 \\
-2uv & -v^2
\end{bmatrix}
\]

\[
\Rightarrow |J| - J = 0
\]

\[
\Rightarrow \lambda^2 - (-u_e^2 - 1 + 2u_e v_e) \lambda + u_e^2 = 0
\]

For \((u_e, v_e)\) to be an unstable node or focus, we need:

\[-u_e^2 - 1 + 2u_e v_e > 0\]

\[-(a+b)^2 - 1 + 2(a+b) \frac{b}{(a+b)^2} > 0\]

\[-(a+b)^2 - 1 + \frac{2b}{a+b} > 0\]
(i) No, because the positively invariant region must include the boundary of the region bounded by these trajectories, and thus this positively invariant set contains 4 equilibria. Thus Poincaré–Bendixson cannot be used.

(ii) Consider \( V(x, y) = -\frac{1}{2} (x^2 + y^2) + \frac{1}{2} (xy^2 - \frac{x^3}{3}) \)

\[
\frac{\partial V}{\partial x} = -x + \frac{y^2}{2} - \frac{x^2}{2}
\]

\[
\frac{\partial V}{\partial y} = -y + xy
\]

\[ V(x, y) = 0 \]

\[ \Rightarrow V(x, y) = \text{const. are } \text{trajectories of the system} \]

Consider \( V(x, y) \) near the origin.

\[ V(x, y) = \text{const.} \]

\[ \Rightarrow -\frac{1}{2} (x^2 + y^2) + \frac{1}{2} (xy^2 - \frac{x^3}{3}) = \text{const} \]

Level sets of \( V(x, y) \) for small \( x, y \), i.e., for

\[ x = \varepsilon_1, \ y = \varepsilon_2 \]

\[ \Rightarrow \varepsilon_1^2 + \varepsilon_2^2 = \text{const.} \]

\[ \Rightarrow \text{closed orbits around origin} \]

and, since the region is invariant, these closed orbits are contained in this region.
2. \[ \dot{x} = a_1 x - a_2 x^2 - a_3 b x \]
\[ \Rightarrow \dot{x}_1 = (a_1 a - a_3 b) x - a_2 x^2 \]

equilibria:
\[ \dot{x} = 0 \Rightarrow (a_2 - a_3 b) x_e = a_2 x_e^2 \]
\[ \Rightarrow x_e = 0 \text{ or } x_e = \frac{a_1 a - a_3 b}{a_2} = \mu \]

\[ D_x f_{\mu} \mid_{x_e} = \mu - 2 a_2 x_e \]
\[ \Rightarrow \text{Bifurcation possible at } x_e = 0, \mu = 0. \]

\[ x_e = 0: D_x f_{\mu} \mid_{x_e} = \mu \]
\[ \Rightarrow \mu > 0: \text{Unstable equilibrium at } x_e = 0 \]
\[ \mu < 0: \text{Stable} \]

\[ x_e = \frac{\mu}{a_2}: D_x f_{\mu} = \mu - 2 a_2 \frac{\mu}{a_2} = -\mu \]
\[ \Rightarrow \mu > 0: \text{Stable equilibrium at } x = \frac{\mu}{a_2} \]
\[ \mu < 0: \text{Unstable} \]

Transcritical bifurcation:

![Transcritical bifurcation diagram](image-url)
3. Note that $N(a)$ is real and so $-\frac{1}{N(a)}$ lies on the negative real axis.

Now $G(jw) = \frac{(1-w^2-2jw)(4-w^2-4jw)}{(1+w^2)^2(4+w^2)^2}$

$\text{Im}(G(jw)) = -\frac{2w(6-3w^2)}{x} = 0$

$\Rightarrow \quad w = \sqrt{2}$

$\therefore \quad 1 + N(a) \text{ Re}(G(j\sqrt{2})) = 0$

Check that $N(a) = 18$

Because $N(a)$ starts from $b$ at $a = 0$ and decreases after $a = \frac{a}{b}$, the equation $N(a) = 18$ has a solution if $b > 18$.

The frequency of oscillation will be close to $w = \sqrt{2}$.

4. $\dot{x}_1 = -x_1^3$, $\dot{x}_2 = x_1$ equilibria at $x_1 = 0, x_2 \in \mathbb{R}$

$\begin{bmatrix} -3x_1^2 & 0 \\ 1 & 0 \end{bmatrix} x_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ Hartman–Grobman cannot be used

We can go back and look at the equations directly! The origin (or any point in $\mathbb{R}^2$) is not SSS for this system.

Proof: Given $x_0 = (s,0)$ for any $s > 0$ the

$\text{Ensuring } x_2(t; x_0, 0) \to \infty$.

$\therefore \exists \epsilon > 0$ such that $\|x(0) - x_0\| < \epsilon \Rightarrow \|x(t) - x_e\| < \epsilon$ for all