Stanford University

ENGR 76

Information Science and Engineering

Lecture 10: Sampling and Interpolation

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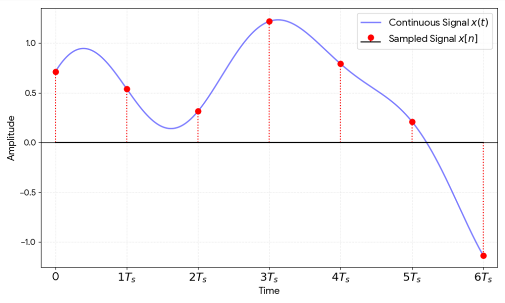
Recap

Sampling

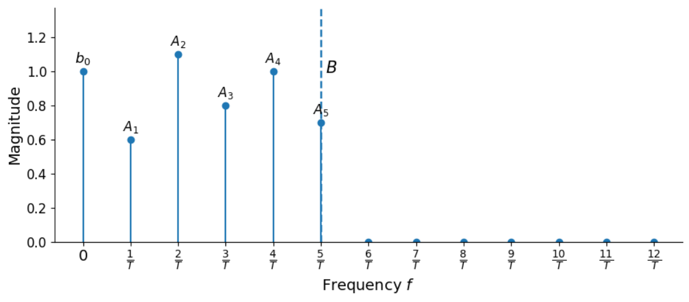


- Given continuous time signal $x(t)$
  - Sampling it at uniform intervals of T_s seconds
  - Obtain discrete-time sequence $x(0), x(T_s), x(2T_s), \ldots$

Baseband Signal

Definition (Baseband Signal)

A signal is baseband if the Fourier series representation contain only those components which have frequencies in $[0, B]$, i.e. A_j is non-zero only if $\frac{j}{T} \leq B$ (or equivalently $A_j = 0$ if $j/T > B$).



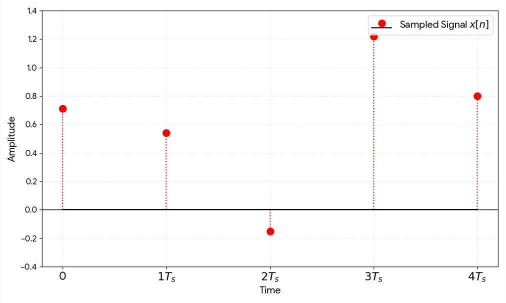
Shannon-Nyquist Sampling Theorem

Theorem

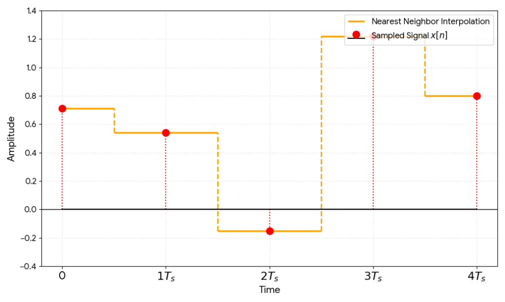
Let $x(t)$ be a baseband signal with bandwidth B. Then $x(t)$ can be perfectly reconstructed from its samples $x(kT_s)$ for $k \in \mathbb{Z}$ if $T_s < 1/(2B)$ or $f_s > 2B$.

Interpolation

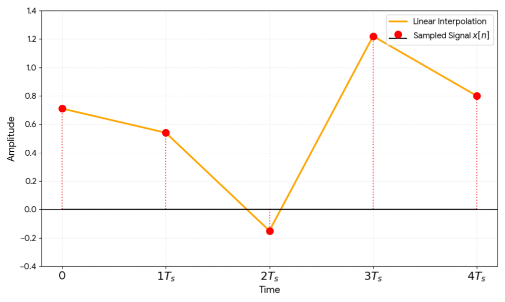
- Process of constructing continuous-time signal from discrete samples by filling in the values between the sampling instants



Nearest Neighbor Interpolation



Linear Interpolation



Perfect Reconstruction

- Suppose we have a bandlimited signal $x(t)$ with band $[0, B]$
- Can either $\hat{x}_{nn}(t)$ or $\hat{x}_{lin}(t)$ be a perfect reconstruction?
  - No!
  - Nearest neighbor interpolation has jump discontinuities
  - Linear interpolation has corners
  - Both of these require infinite frequency components
  - But the original signal was bandlimited
- What interpolation gives perfect reconstruction?
- General interpolation function?

Interpolation

Interpolation Function

- Interpolation is deciding for each t, how much sample mT_s contributes
- Nearest neighbor: Sample at mT_s has box-shaped influence for time in $\left[mT_s - \frac{T_s}{2}, mT_s, \frac{T_s}{2}\right]$
- Linear: Sample at mT_s has triangle-shaped influence for time in $[mT_s - T_s, mT_s + T_s]$
- General interpolation function: deciding this influence...

Interpolated Signal

An interpolated signal $\hat{x}(t)$ is obtained from samples $\{x(0), x(T_s), x(2T_s), \ldots\}$ using

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s) F\left(\frac{t - mT_s}{T_s}\right),$$

where $F(t)$ is a valid interpolation function if

- $F(0) = 1$
- $F(k) = 0$ for all non-zero integers k

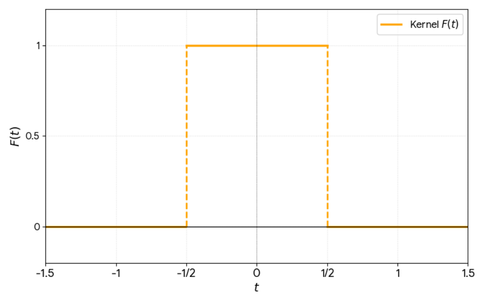
What are we doing?

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s) F\left(\frac{t - mT_s}{T_s}\right)$$

- Deciding the value of $\hat{x}(t)$
  - Influence at time t of sample at mT_s
    - $F\left(\frac{t-mT_s}{T_s}\right)$
    - Scaled by the value of sample $x(mT_s)$
  - Summing up the influence from all samples
  - Gives $\hat{x}(t)$

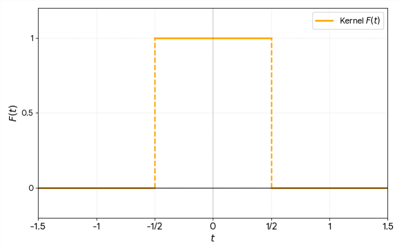
Example

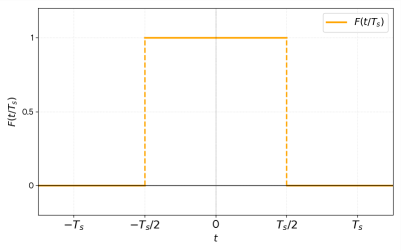
$$F(t) = \begin{cases} 1, & -\frac{1}{2} \leq t \leq \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$



What does $g(t) = F\left(\frac{t}{T_s}\right)$ look like?

Example

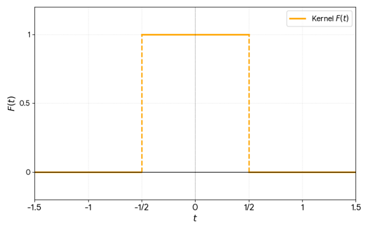


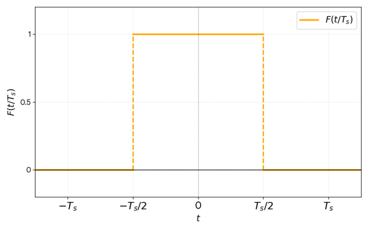


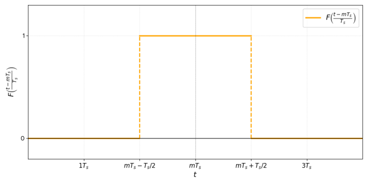
$F\left(\frac{t}{T_s}\right)$ is just a stretched version

What does $h(t) = F\left(\frac{t - mT_s}{T_s}\right)$ look like?

Example





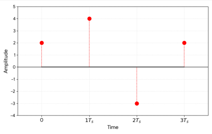


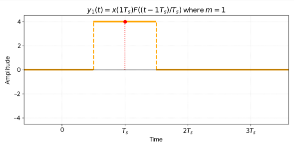
$F\left(\frac{t-mT_s}{T_s}\right)$ is just shifted to mT_s

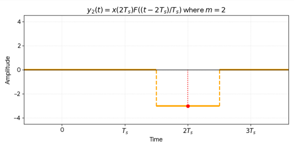
Example

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s)F\left(\frac{t - mT_s}{T_s}\right)$$

- Define $y_m(t) = x(mT_s)F\left(\frac{t-mT_s}{T_s}\right)$
    - Contribution of m-th sample

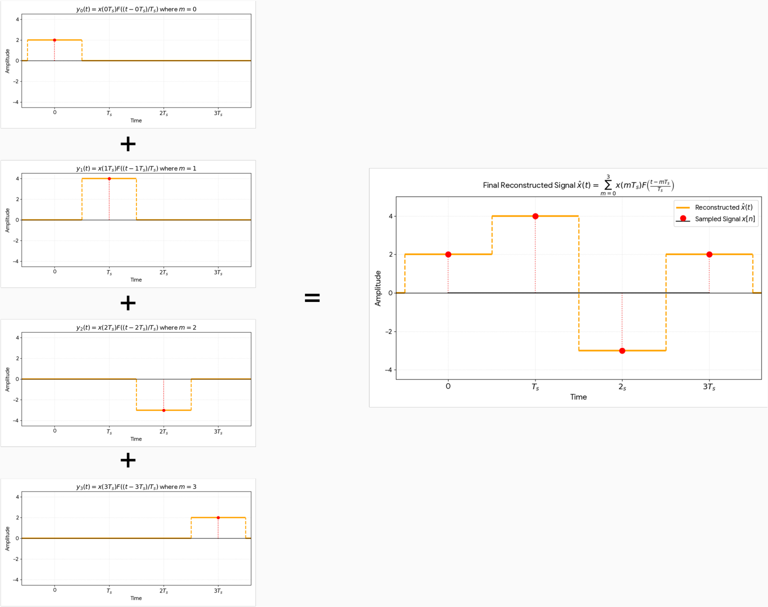




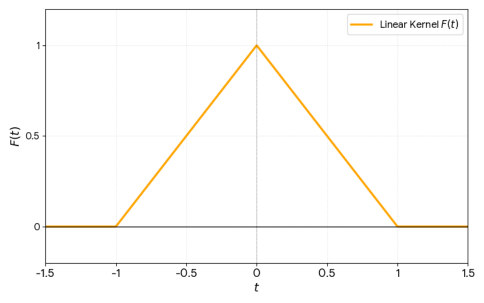


$$\hat{x}(t) = \sum_{m=\infty}^{\infty} y_m(t)$$

Example


$y_0(t) = x(0T_s)F((t - 0T_s)/T_s)$ where $m = 0$
$y_1(t) = x(1T_s)F((t - 1T_s)/T_s)$ where $m = 1$
$y_2(t) = x(2T_s)F((t - 2T_s)/T_s)$ where $m = 2$
$y_3(t) = x(3T_s)F((t - 3T_s)/T_s)$ where $m = 3$
Amplitude
Time
0
T_s
$2T_s$
$3T_s$
+
=
Final Reconstructed Signal $\hat{x}(t) = \sum_{m=0}^{3} x(mT_s)F\left(\frac{t - mT_s}{T_s}\right)$
Reconstructed $\hat{x}(t)$
Sampled Signal $x[n]$
2_s

Example: Linear Interpolation



Explore this yourself in mini-PSet and practice problems!

Valid Interpolation Function

- $F(t)$ is a valid interpolation function if
  - $F(0) = 1$
  - $F(k) = 0$ for all non-zero integers k
- Linked to our basic requirement that $\hat{x}(mT_s) = x(mT_s)$
  - Reconstructed function should match the samples at sampling instances

Valid Interpolation Function

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s) F\left(\frac{t - mT_s}{T_s}\right)$$

What happens at sampling instances? For example at $t = 2T_s$:

$$\begin{aligned}\hat{x}(2T_s) &= \sum_{m=-\infty}^{\infty} x(mT_s) F\left(\frac{2T_s - mT_s}{T_s}\right) \\ &= \sum_{m=-\infty}^{\infty} x(mT_s) F\left(2 - m\right) \\ &= x(0)F(2) + x(T_s)F(1) + x(2T_s)F(0) + x(3T_s)F(-1) + \ldots\end{aligned}$$

- Require $F(0) = 1$ and $F(k) = 0$ for non-zero integer k

Sinc Interpolation

Polynomial Interpolation Function

- Let us try to create a polynomial interpolation function
- $F(t) = (1 - t)(1 + t)$
    - Is this valid?

Polynomial Interpolation Function

- Let us try to create a polynomial interpolation function
- $F(t) = (1-t)(1+t)$
  - Not valid
  - $F(2) = -3$
- Keep adding roots at all integer points!

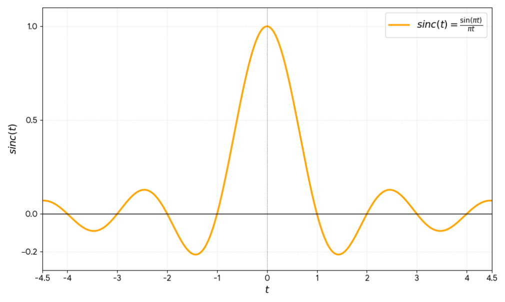
$$F(t) = \left(1-t\right)\left(1+t\right)\left(1-\frac{t}{2}\right)\left(1+\frac{t}{2}\right)\left(1-\frac{t}{3}\right)\left(1+\frac{t}{3}\right)\ldots$$
$$= \mathsf{sinc}(t)$$

- **Sinc Interpolation**

Sinc Interpolation

$$\text{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

- At $t = 0$, defined as $\text{sinc}(0) = 1$
- $\text{sinc}(k) = 0$ for all non-zero integer k
- $\text{sinc}(t)$ is non-zero for all non-integer t



Shannon-Nyquist Sampling Theorem

Fact

Let $x(t)$ be a baseband signal with spectrum supported on $[0, B]$. Suppose the sampling frequency satisfies $f_s > 2B$. Then $x(t)$ can be perfectly reconstructed from its samples $\{x(mT_s)\}_{m\in\mathbb{Z}}$ using

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s)\,\mathrm{sinc}\left(\frac{t - mT_s}{T_s}\right).$$

That is, if $f_s > 2B$, then sinc interpolation yields perfect reconstruction, i.e., $\hat{x}(t) = x(t)$ for all t.

Dependence on Sampling Frequency

Sampling Frequency

- Visualization to understand how reconstructions change with varying frequency

Sampling Frequency and Sinc Interpolation

- Output of sinc interpolation is bandlimited based on the sampling frequency

Sinc Interpolation

Fact

Let $x(t)$ be bandlimited to $[0, B]$, and let

$$\hat{x}(t) = \sum_{m=-\infty}^{\infty} x(mT_s)\,\mathrm{sinc}\left(\frac{t - mT_s}{T_s}\right).$$

Then $\hat{x}(t)$ is bandlimited to $\left[0, \frac{f_s}{2}\right]$. Moreover,

- If $f_s > 2B$, then $\hat{x}(t) = x(t)$ for all t.
- If $f_s < 2B$, then $\hat{x}(t) \neq x(t)$, but $\hat{x}(mT_s) = x(mT_s)$.

Stroboscopic Effect

- Continuously moving object looks 'distorted' because we are observing it at discrete time instants
    - Slowed down, frozen or moving backwards
- Visualization on Stroboscopic effect

Thank You!