# Lecture 5-8: Fourier Analysis and Spectral Representation of Signals

*ENGR 76 lecture notes* — Updated: April 19, 2025
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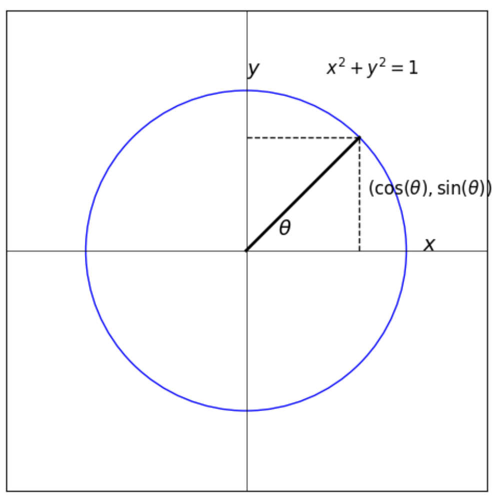
## 1 Sine and Cosine

This week, we investigate what we will (earnestly) refer to as a Major Secret of the Universe:

Every signal has a spectrum, and the spectrum determines the signal.

Learning this secret is about learning Fourier analysis. The subject is both vast and deep, and its origins going back to the French mathematician and physicist Jean-Baptiste Joseph Fourier, 1822, are far removed from where we are headed. In this course, we will see three different applications of Fourier analysis: 1) compression of images; 2) analog to digital conversion; 3) design of digital communication systems.

We start by recalling that $\sin(\theta)$ and $\cos(\theta)$ can be defined as the $x$ and $y$ coordinates of a point on the unit circle that lies at angle $\theta$ radians from the $x$-axis. See the picture below.




This definition can be regarded as an extension of the more elementary definitions of sine and cosine on a right triangle that you may have first encountered in your studies. It allows us to define sine and cosine for angles beyond $[0, \pi/2]$, including negative angles and immediately observe the following relationships:

$$\cos(\theta + 2\pi k) = \cos(\theta) \qquad \sin(\theta + 2\pi k) = \sin(\theta) \quad \text{for} \quad k = 0, \pm 1, \pm 2, \ldots$$

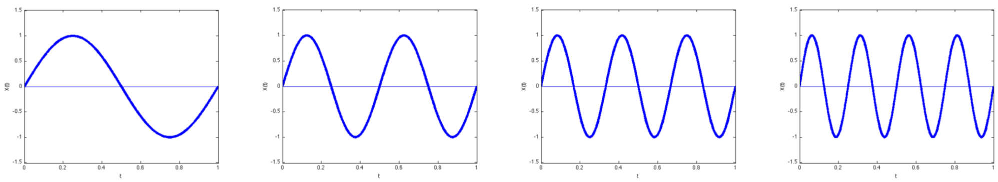
From this interpretation of $\sin(\theta)$ and $\cos(\theta)$ as giving a point on the unit circle, we move toward a dynamic picture and think of this point as moving on the circle in the counterclockwise direction. Assume the point completes $f$ cycles around the circle in 1 second, so that the angle $\theta$ now becomes a linear function of time: $\theta = 2\pi f t$. The $x$ and $y$ coordinates of the point are now functions of time $t$ and are given by

$$\cos(2\pi f t) \qquad \sin(2\pi f t) \quad \text{for} \quad t \in \mathbb{R}.$$

More generally, we will allow ourselves the flexibility to start from an arbitrary initial point, specified by a phase $\phi$, and modify the radius of the circle to $A$ to obtain the functions

$$A\cos(2\pi f t+\phi) \qquad A\sin(2\pi f t+\phi) \quad \text{for} \quad t\in\mathbb{R}. \tag{1}$$

The number $A$ is called the amplitude and $f$ is called the frequency. The below plots depict $\sin(2\pi f t)$ for $f=1$, $f=2$ and $f=3$:



The frequency has units cycles per second and dimension 1/sec. This unit is called Hertz, abbreviated Hz, after Heinrich Hertz (1857-1894), who experimentally demonstrated the existence of electromagnetic waves verifying the theoretical work of James Clerk Maxwell.

Recall that a function $f$ is called periodic if there exists a constant $T$ such that

$$f(t+T)=f(t), \qquad \forall t\in\mathbb{R}.$$

The smallest $T>0$ for which the function satisfies this property is called the period of the function. Note that the functions in (1) are periodic with period

$$T=\frac{1}{f}.$$

## 2 It all adds up: Fourier Series Representation of Periodic Signals

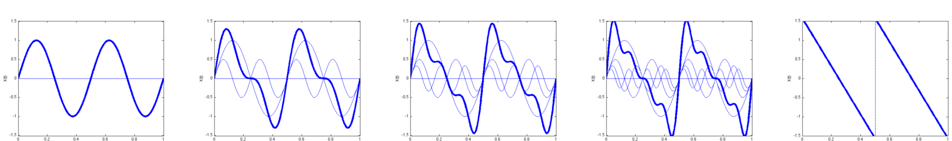
By adding sinusoids of the form

$$\sum_{j=1}^{N} A_j \sin(2\pi \frac{j}{T} t+\phi_j),$$

we can approximate other interesting periodic functions with period $T$. For example, the sawtooth function of period 0.5 can be written in the form

$$\begin{aligned} y(t) &= \sum_{j=1}^{\infty} \frac{1}{j}\sin(4\pi\, j\, t), \\ &= \sin(4\pi t)+\frac{1}{2}\sin(8\pi t)+\frac{1}{3}\sin(12\pi t)+\frac{1}{4}\sin(16\pi t)+\dots \end{aligned}$$

The images below illustrate the individual sinusoidal components and how they combine to approximate the sawtooth function. The first image shows the contribution of the first term alone; the second shows the sum of the first two terms; the third includes the first three terms; the fourth includes the first four terms. The final image shows the infinite sum, which exactly reconstructs the original sawtooth function.



Note that in order to obtain a periodic function of period $T$, we include sinusoids that complete an integer number of cycles within the interval of length $T$. Specifically, we start with the smallest (fundamental) frequency $f_1 = \frac{1}{T}$, which corresponds to one cycle in $T$ seconds, and add the higher frequency harmonics $f_j = \frac{j}{T}$, where the $j$'th harmonic completes $j$ cycles within the length $T$ interval.

Joseph Fourier's major secret of the universe is that any (well-behaved) periodic signal $y(t)$ with period $T$ can be written in this form, i.e.,

$$y(t) = b_0 + \sum_{j=1}^{\infty} A_j \sin(2\pi \frac{j}{T} t + \phi_j), \tag{2}$$

where we add a constant term $b_0$, which can be regarded as the zero frequency component. This representation can be rewritten in the following alternative form by using the trigonometric identity $\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$:

$$\begin{aligned} y(t) &= b_0 + \sum_{j=1}^{\infty} A_j \cos\phi_j \sin(2\pi \frac{j}{T} t) + A_j \sin\phi_j \cos(2\pi \frac{j}{T} t), && (3) \\ &= b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t), && (4) \end{aligned}$$

where we define $a_j = A_j \cos\phi_j$ and $b_j = A_j \sin\phi_j$. Note that the two representations in (2) and (4) are equivalent; in both representations we have two parameters associated with each frequency component (you can think of them as knobs we can adjust to control the contribution of each frequency component). We can easily go back and forth between (2) and (4) using $a_j = A_j \cos\phi_j$ and $b_j = A_j \sin\phi_j$ to go to (4) from (2), and $A_j^2 = a_j^2 + b_j^2$ and $\tan\phi_j = b_j/a_j$ to go to (2) from (4).

How can we compute the coefficients $a_k$ and $b_k$ of the $k$'th harmonic in (4) for a given signal $y(t)$ ? These coefficients can be computed by the following formulas:

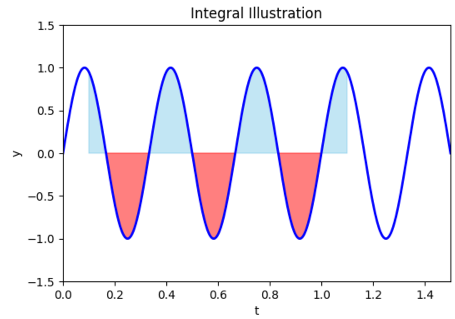
$$\begin{aligned} a_k &= \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt \qquad k = 1, 2, 3, \dots && (5) \\ b_k &= \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\pi \frac{k}{T} t) dt. \qquad k = 1, 2, \dots && (6) \\ b_0 &= \frac{1}{T} \int_{-T/2}^{T/2} y(t) dt && (7) \end{aligned}$$

Assuming $y(t)$ can be written in the form (4), it is not too difficult to verify that the above formulas extract the desired coefficients. For example, plug in the expression for $y(t)$ in (4) in the formula for $a_k$ for some $k = 1, 2, 3, \dots$:

$$\begin{aligned} \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt &= \frac{2}{T} \int_{-T/2}^{T/2} \left( b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t) \right) \sin(2\pi \frac{k}{T} t) dt && (8) \\ &= \frac{2}{T} \int_{-T/2}^{T/2} b_0 \sin(2\pi \frac{k}{T} t) dt && (9) \\ &+ \sum_{j=1}^{\infty} \frac{a_j}{T} \int_{-T/2}^{T/2} 2 \sin(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt && (10) \\ &+ \sum_{j=1}^{\infty} \frac{b_j}{T} \int_{-T/2}^{T/2} 2 \cos(2\pi \frac{j}{T} t) \sin(2\pi \frac{k}{T} t) dt. && (11) \end{aligned}$$

Now observe that if you take the sine or the cosine function and integrate it over one period, the integral will be equal to zero. This is because of the symmetry between the positive and negative parts of the function:

the contribution to the integral from the parts where the function takes positive values will cancel exactly with the contribution from the parts where the function takes negative values. This conclusion also holds when integrating the function over any interval whose length is an integer multiple of the period. See the picture below:




This implies that the first term (9) above is equal to zero:

$$\frac{2}{T}\int_{-T/2}^{T/2} b_0 \sin(2\pi\frac{k}{T}t)dt = 0.$$

The terms in the summation in (10) can be rewritten as

$$\frac{a_j}{T}\int_{-T/2}^{T/2} 2\sin(2\pi\frac{j}{T}t)\sin(2\pi\frac{k}{T}t)dt = \frac{a_j}{T}\int_{-T/2}^{T/2}\cos(2\pi\frac{j-k}{T}t)dt - \frac{a_j}{T}\int_{-T/2}^{T/2}\cos(2\pi\frac{j+k}{T}t)dt,$$

where we used the trigonometric identity $2\sin\alpha\sin\beta = \cos(\alpha-\beta) - \cos(\alpha+\beta)$. Note that by the same argument, both of the integrals above will be equal to zero when $j \neq k$, since we are integrating a cosine function over an interval of length $T$ and the function completes an integer number of cycles ($j-k$ or $j+k$ respectively) in this interval. The only non-zero term corresponds to $j = k$, in which case we have

$$\begin{aligned}\frac{a_k}{T}\int_{-T/2}^{T/2} 2\sin(2\pi\frac{k}{T}t)\sin(2\pi\frac{k}{T}t)dt &= \frac{a_k}{T}\int_{-T/2}^{T/2}\cos(2\pi\frac{0}{T}t)dt - \frac{a_j}{T}\int_{-T/2}^{T/2}\cos(2\pi\frac{2k}{T}t)dt \\ &= \frac{a_k}{T}\int_{-T/2}^{T/2} 1\,dt \\ &= a_k,\end{aligned}$$

where we used the fact that $\cos(0) = 1$. Similarly one can argue that all the terms in (11)

$$\begin{aligned}&\frac{b_j}{T}\int_{-T/2}^{T/2} 2\cos(2\pi\frac{j}{T}t)\sin(2\pi\frac{k}{T}t)dt \\ &\qquad = \frac{b_j}{T}\int_{-T/2}^{T/2}\sin(2\pi\frac{j+k}{T}t)dt + \frac{b_j}{T}\int_{-T/2}^{T/2}\sin(2\pi\frac{j-k}{T}t)dt \\ &\qquad = 0.\end{aligned}$$

This shows that the expression in (8) indeed extracts the coefficient $a_k$ in the representation of $y(t)$ as desired.

We also note that $\cos(2\pi\frac{j}{T}t) = 1$ for $j = 0$, and therefore

$$b_0 = \frac{1}{T}\int_{-T/2}^{T/2} y(t)dt,$$

i.e., the constant term $b_0$ corresponds to the average value of the signal.

# 3 Connection to Linear Algebra

Assume the function $y(t)$ is zero-mean, i.e. the constant term $b_0$ in the representation in (4) is equal to zero. Then the Fourier series representation reduces to:

$$y(t) = \sum_{k=0}^{\infty} a_k \cos(2\pi \frac{k}{T} t) + b_k \sin(2\pi \frac{k}{T} t), \tag{12}$$

with the coefficients in this representation given by the formula

$$\begin{aligned} a_k &= \frac{2}{T} \int_0^T y(t) \sin(2\pi \frac{k}{T} t) dt \qquad k = 1, 2, 3, \ldots \\ b_k &= \frac{2}{T} \int_0^T y(t) \cos(2\pi \frac{k}{T} t) dt. \qquad k = 1, 2, \ldots \end{aligned}$$

If you are familiar with the notions of an inner product space and orthonormal basis from linear algebra, one way to interpret (12) is by thinking of the set of functions $\{e_1(t) = \sin(2\pi \frac{1}{T} t), e_2(t) = \cos(2\pi \frac{1}{T} t), e_3(t) = \cos(2\pi \frac{2}{T} t), e_4(t) = \cos(2\pi \frac{2}{T} t), \ldots\}$ in (12) as forming an orthonormal basis for the inner product space of (well-behaved) periodic functions with period $T$, with the inner product between any two functions in this space defined as:

$$< v(t), w(t) >= \frac{2}{T} \int_0^T v(t) w(t) dt. \tag{13}$$

With this interpretation, (12) simply states that any function $y(t)$ in this space can be expressed as a linear combination of the basis functions $\{e_1(t), e_2(t), e_3(t), \ldots\}$. In order to find the coefficients in this representation we need to compute the inner product of the function $y(t)$ with the basis functions $< y(t), e_k(t) >$ for $k = 1, 2, \ldots$ according to (13), the defintion of inner product in this space. This is analogous to how we can represent a vector $\mathbf{v} \in \mathbb{R}^n$ in terms of an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ in the form:

$$\mathbf{v} = \sum_{k=1}^{n} < \mathbf{v}, \mathbf{e}_k > \mathbf{e}_k,$$

i.e. $\mathbf{v}$ is decomposed as a sum of vectors in the directions of the orthonormal basis vectors, and the components are given by the (usual) inner product of $\mathbf{v}$ with the basis vectors.

# 4 Fourier Cosine Series

In this section, we will investigate what happens when we apply the Fourier series decomposition in (4) to an even function. Recall that a function $f(t)$ is called even if $f(-t) = f(t)$ for all $t \in \mathbb{R}$. For example the cosine function is even since $\cos(-t) = \cos(t)$. In constrast, a function $h(t)$ is called odd if $h(-t) = -h(t)$ for all $t \in \mathbb{R}$. The sine function is odd since $\sin(-t) = -\sin(t)$. Note that the product of an even function $f(t)$ and an odd function $h(t)$ is odd since

$$f(-t)h(-t) = f(t)\,(-h(t)) = -f(t)h(t).$$

What happens when we compute the Fourier series decomposition in (4) for an even function? Assume $y(t)$ is even. Using the formula in (5), coefficients $a_k$ are given by

$$a_k = \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{k}{T} t) dt.$$

Note that the product function $y(t) \sin(2\pi \frac{k}{T} t)$ is odd because it is the product of an even function $y(t)$ and and odd function $\sin(2\pi \frac{k}{T} t)$. Note that the integral of an odd function over an interval of the form

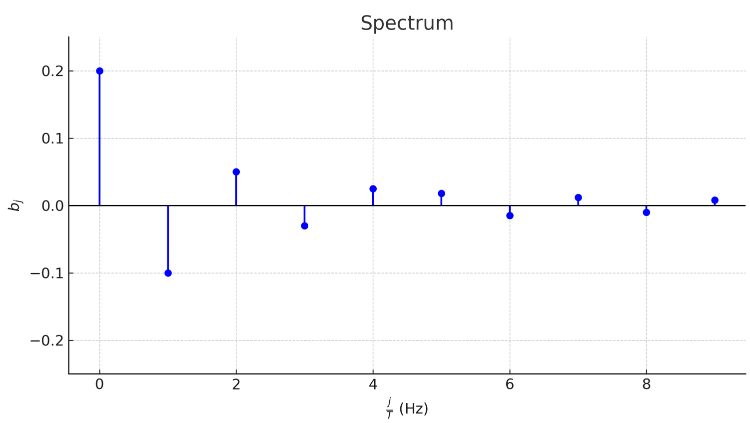
$[-T/2, T/2]$ is always zero because the intergral over the $[-T/2, 0]$ cancels the integral over the duration $[0, T/2]$. Formally,

$$\begin{aligned}
a_k &= \frac{2}{T}\int_{-T/2}^{T/2} y(t)\sin(2\pi\frac{k}{T}t)dt \\
&= \frac{2}{T}\int_{-T/2}^{0} y(t)\sin(2\pi\frac{k}{T}t)dt + \frac{2}{T}\int_{0}^{T/2} y(t)\sin(2\pi\frac{k}{T}t)dt \\
&= -\frac{2}{T}\int_{0}^{T/2} y(t)\sin(2\pi\frac{k}{T}t)dt + \frac{2}{T}\int_{0}^{T/2} y(t)\sin(2\pi\frac{k}{T}t)dt \\
&= 0.
\end{aligned}$$

This shows that the Fourier series representation of an even function $y(t)$ has the following simpler form which contains only cosine terms:

$$y(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi\frac{j}{T}t).$$

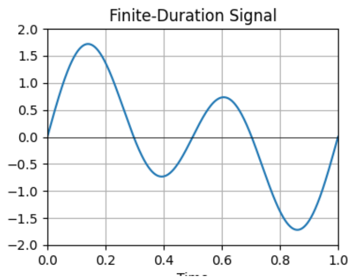
This is sometimes refered to as the Fourier cosine series representation of an even function. We can visualize the Fourier cosine series representaiton of a signal (with $T = 1$) in a plot as shown below which we will refer to as the spectrum of the signal.

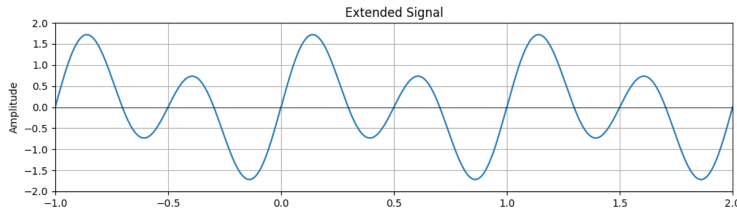



This notion will be useful in the next chapter, when we look into developing a Fourier series representation for finite duration signals.

# 5 Finite-duration Signals

How about when the signal is not periodic? Many of the signals we will encounter in practice will be finite-duration signals which are only defined in a finite interval of duration $T$. For such finite duration signals, we can construct their so-called periodic extension, which is a periodic signal with period $T$ equal to the finite-duration signal, and compute the Fourier series representation of the periodic extension. See the figure below.



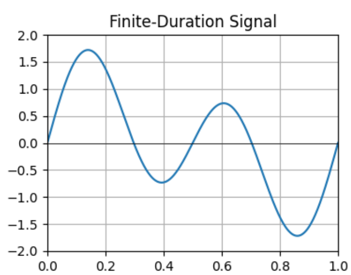



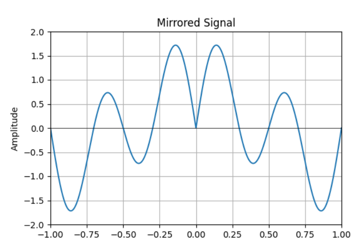

The periodic extension in the lower picture has a Fourier series representation of the form

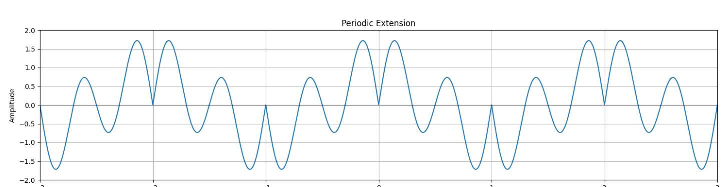
$$y_p(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin(2\pi \frac{j}{T} t) + b_j \cos(2\pi \frac{j}{T} t).$$

We can take that to be the Fourier series representation of the finite-duration signal.

There multiple ways in which we can construct a periodic signal from a finite-duration signal. For example, we could also do the following. We can first take the mirror image of the finite-duration signal with respect to the y-axis to create an even signal of duration $2T$. We can then construct the periodic extension of this even signal of duration $2T$. See the figure below.



The periodic signal in the third image is called the even periodic extension of the finite duration signal in the first image. Note that because the even periodic extension of the signal is even and periodic with period $2T$, it has a Fourier series representation of the form

$$y(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi \frac{j}{2T} t).$$

This is desirable since the Fourier representation is now simpler and only contains cosine terms. Moreover, the even periodic extension of the signal always yields a continious signal while the periodic extension can be discontinious when the signal does not start and end with the same value.

# 6 Infinite-duration non-periodic signals

What happens when the duration of the signal $T \to \infty$? This gives a infinite duration signal that is not necessarily periodic. Note that the gap between consecutive frequencies in the spectrum of a periodic signal is $\frac{1}{T}$. If we consider an infinite duration signal with $T \to \infty$, then we can expect the frequency domain representation to be no longer discrete, and instead include a continuum of frequencies.

With a certain amount of bravado and disregard for mathematical politeness we can see what happens in the limit $T \to \infty$ with the following argument. Write $\frac{1}{T} = \Delta$ and the points $\frac{j}{T} = j\Delta$. Then the Fourier series representation of a periodic signal becomes

$$\begin{aligned}
y(t) &= b_0 + \sum_{j=1}^{\infty} \left( \frac{2}{T} \int_{-T/2}^{T/2} y(t) \sin(2\pi \frac{j}{T} t) dt \right) \sin(2\pi \frac{j}{T} t) + \left( \frac{2}{T} \int_{-T/2}^{T/2} y(t) \cos(2\pi \frac{j}{T} t) dt \right) \cos(2\pi \frac{j}{T} t), \\
&= b_0 + \sum_{j=1}^{\infty} \Delta \left( \int_{-T/2}^{T/2} 2y(t) \sin(2\pi (j\Delta) t) dt \right) \sin(2\pi (j\Delta) t) + \Delta \left( \int_{-T/2}^{T/2} 2y(t) \cos(2\pi (j\Delta) t) dt \right) \cos(2\pi (j\Delta) t)
\end{aligned}$$

Looks like a Riemann sum of the form $\sum_{j=1}^{\infty} \Delta g(j\Delta) \to \int_0^{\infty} g(f) df$ when $\Delta \to 0$. Looks like when $T \to \infty$, the expression becomes

$$y(t) = b_0 + \int_0^{\infty} \left( 2 \left( \int_{-\infty}^{\infty} y(t) \sin(2\pi f t) dt \right) \sin(2\pi f t) + \left( 2 \int_{-\infty}^{\infty} y(t) \cos(2\pi f t) dt \right) \cos(2\pi f t) \right) df.$$

We will turn this into a definition. For any (well-behaved) non-periodic signal $y(t)$, we define its Fourier transform to be given by the following two functions

$$\begin{aligned}
A(f) &= 2 \int_{-\infty}^{\infty} y(t) \sin(2\pi f t) dt, \\
B(f) &= 2 \int_{-\infty}^{\infty} y(t) \cos(2\pi f t) dt.
\end{aligned}$$

The integration is with respect to $t$ but the integrand involves both the variables $t$ and $f$. After integration what remains is a function of $f$. The formula looks similar to that for the Fourier coefficients but unlike the discrete Fourier coefficients for a periodic signal, the Fourier transform of a general (non-periodic) signal is now a (transformed) function of a continuous real variable $f$. We still think of $f$ as a frequency variable, and the Fourier transform functions $A(f)$ and $B(f)$ as the frequency domain representation of the signal $y(t)$, but instead of discrete frequencies as in the periodic case we have a continuum of frequencies in the non-periodic case. The spectrum of $y(t)$ is the set of real numbers $f$ where $A^2(f) + B^2(f) \neq 0$ (or equivalently either $A(f)$ or $B(f)$ is non-zero). The bandwidth of the signal is defined as

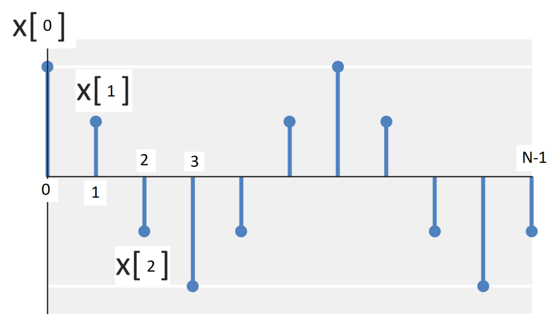
$$B = f_{max} - f_{min},$$

where $f_{max}$ is largest frequency $f$ for which $A^2(f) + B^2(f) \neq 0$ and $f_{min}$ is smallest frequency $f$ for which $A^2(f) + B^2(f) \neq 0$. The signal $y(t)$ can be written in terms of its spectrum by taking an integral over all frequencies $f$:

$$y(t) = b_0 + \int_0^\infty A(f) \sin(2\pi f t) + B(f) \cos(2\pi f t) df.$$

# 7 Discrete Fourier Transform (DFT)

We next consider a signal in discrete time of duration $L$, with values $X[k]$ for $k = 0, \ldots, L-1$, as shown below.




We can view this either as a finite-length signal or, equivalently, as a periodic signal with period $L$, where $X[k+L] = X[k]$ for all $k$. Similar to the continuous-time case, we can represent this signal in terms of its frequency components:

$$X[k] = b_0 + \sum_j a_j \sin\left(2\pi \frac{j}{L} k\right) + b_k \cos\left(2\pi \frac{j}{L} k\right),$$

where the $j$th term in the summation corresponds to the $j$th harmonic—i.e., a sinusoid that completes $j$ cycles over the interval $k = 0, \ldots, L-1$.

In continuous time, $j$ could be arbitrarily large. However, in discrete time we observe that for $i = j + L$,

$$\begin{aligned} \cos\left(2\pi \frac{i}{L} k\right) &= \cos\left(2\pi \frac{j+L}{L} k\right) \\ &= \cos\left(2\pi \frac{j}{L} k + 2\pi k\right) \\ &= \cos\left(2\pi \frac{j}{L} k\right), \end{aligned}$$

where the last step follows from the periodicity of the cosine function. A similar argument holds for sine. This implies that harmonics repeat every $L$ values of $j$, so it suffices to consider a set of $L$ consecutive values of $j$.

By convention, we take $j = 1, \ldots, L$, and for simplicity assume that $L$ is odd. We can reduce the number of coefficients further by observing that the harmonics for $j = \frac{L+1}{2}, \ldots, L-1$ are redundant—they coincide with the harmonics for $j = 1, \ldots, \frac{L-1}{2}$. To see this, let

$$j = i + \frac{L-1}{2}, \quad \text{for } i = 1, \ldots, \frac{L-1}{2}.$$

Then,

$$\begin{aligned}\cos\left(2\pi\frac{j}{N}k\right) &= \cos\left(2\pi\frac{i+\frac{L-1}{2}}{L}k\right)\\ &= \cos\left(2\pi\frac{i+\frac{L-1}{2}}{L}k - 2\pi k\right)\\ &= \cos\left(2\pi\frac{i-\frac{L+1}{2}}{L}k\right)\\ &= \cos\left(2\pi\frac{\frac{L+1}{2}-i}{L}k\right),\end{aligned}$$

which shows that the $i+\frac{N-1}{2}$th harmonic coincides with the $\frac{N+1}{2}-i$th harmonic. A similar conclusion holds for sine.

Moreover, the $L$th harmonic coincides with the constant term $b_0$, since

$$\sin\left(2\pi\frac{L}{L}k\right) = \sin(2\pi k) = 0, \quad \cos\left(2\pi\frac{L}{L}k\right) = \cos(2\pi k) = 1.$$

Thus, the full representation simplifies to:

$$X[k] = b_0 + \sum_{j=1}^{\frac{L-1}{2}} a_j \sin\left(2\pi\frac{j}{L}k\right) + b_j \cos\left(2\pi\frac{j}{L}k\right).$$

This expression has $1 + 2 \times \frac{L-1}{2} = L$ coefficients, exactly matching the number of values $X[k]$ for $k = 0, \ldots, L-1$. Hence, the coefficients $\{a_j\}, \{b_j\}$ can be obtained by solving a system of $L$ linear equations in $L$ unknowns. The process of going from $(X[0], X[1], \ldots, X[L-1])$ to $\{a_j\}, \{b_j\}$ is called the **Discrete Fourier Transform (DFT)**. The DFT coefficients $b_0$, $a_j$, and $b_j$ for $j = 1, \ldots, \frac{L-1}{2}$ can be computed explicitly by projecting the discrete singal $X[k]$ onto each discrete basis function and proper normalization:

$$b_0 = \frac{1}{L}\sum_{k=0}^{L-1} X[k],$$

$$a_j = \frac{2}{L}\sum_{k=0}^{L-1} X[k] \sin\left[2\pi\frac{j}{L}k\right],$$

$$b_j = \frac{2}{L}\sum_{k=0}^{L-1} X[k] \cos\left[2\pi\frac{j}{L}k\right],$$

for each $j = 1, \ldots, \frac{L-1}{2}$.

The DFT can be computed more efficiently than by directly solving the system of equations, thanks to the highly structured nature of the transformation matrix (involving sines and cosines). This leads to a widely used algorithm known as the **Fast Fourier Transform (FFT)**, which exploits this structure to reduce computational complexity from $\mathcal{O}(L^2)$ to $\mathcal{O}(L \log L)$.

# 8 Discrete Cosine Transform (DCT)

While the Discrete Fourier Transform (DFT) provides a complete frequency-domain representation of a discrete signal using both sine and cosine components, many practical applications—especially in signal compression and image processing—favor a transform that uses only cosine functions. This leads us to the **Discrete Cosine Transform (DCT)**.

The DCT is closely related to the DFT but applies to finite-length signals that are implicitly extended in a symmetric way. Specifically, instead of assuming periodic extension (as in the DFT), the DCT assumes an even reflection of the signal about the boundaries obtaining an even discrete signal of twice the length of the original signal. This subtle difference leads to basis functions that are purely cosines, and typically results in more efficient representations for signals with smooth variations at the boundaries.

Let us assume we are given a discrete signal of length $L$, i.e., a sequence of real numbers:

$$x[0],\ x[1],\ \ldots,\ x[L-1].$$

The DCT represents this signal as a weighted sum of cosine functions:

$$x[k] = \sum_{j=0}^{L-1} b_j \cos\left[\frac{\pi}{L}\left(k + \frac{1}{2}\right) j\right], \quad k = 0, \ldots, L-1.$$

Here, each basis function corresponds to a sampled cosine wave of increasing frequency, and the index $j$ determines the number of half-cycles completed over the interval $k = 0, \ldots, L-1$. Note that the shift by $\frac{1}{2}$ in $k$ ensures the even symmetry of the discrete cosine basis functions matches the symmetry of the even extension of the original signal.

Just like the DFT, the DCT can be understood as a change of basis from the time domain to the frequency domain, but using only basis of cosines instead of mixed sine/cosine terms. The coefficients $b_j$ can be obtained by projecting $x[k]$ onto each basis function and proper normalization:

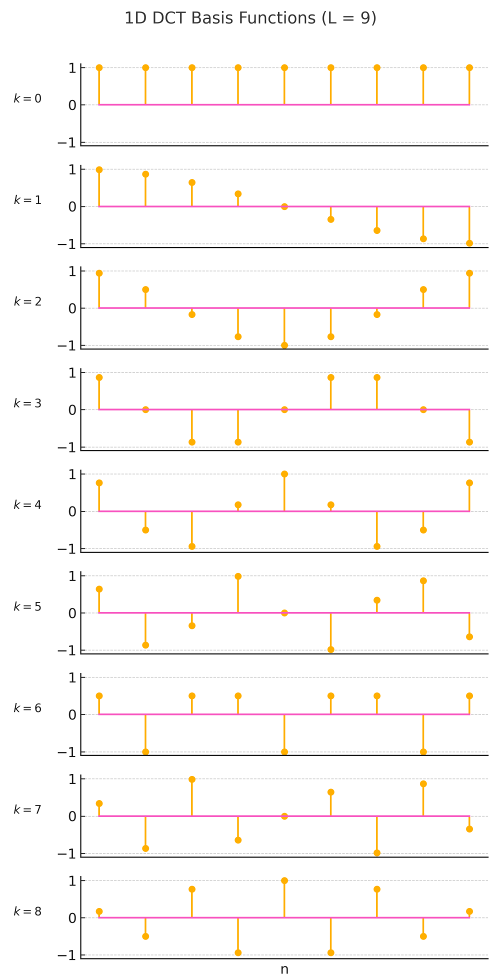
$$b_0 = \frac{1}{L} \sum_{k=0}^{L-1} x[k],$$

$$b_j = \frac{2}{L} \sum_{k=0}^{L-1} x[k] \cos\left(\frac{\pi}{L}\left(k + \frac{1}{2}\right) j\right), \quad \text{for } j = 1, \ldots, L-1.$$

These basis functions can be visualized as discrete samples of cosine waves:

- $j = 0$ corresponds to a constant function.
- Larger values of $j$ correspond to higher-frequency oscillations.

Below is a depiction of the DCT basis functions for $L = 9$:



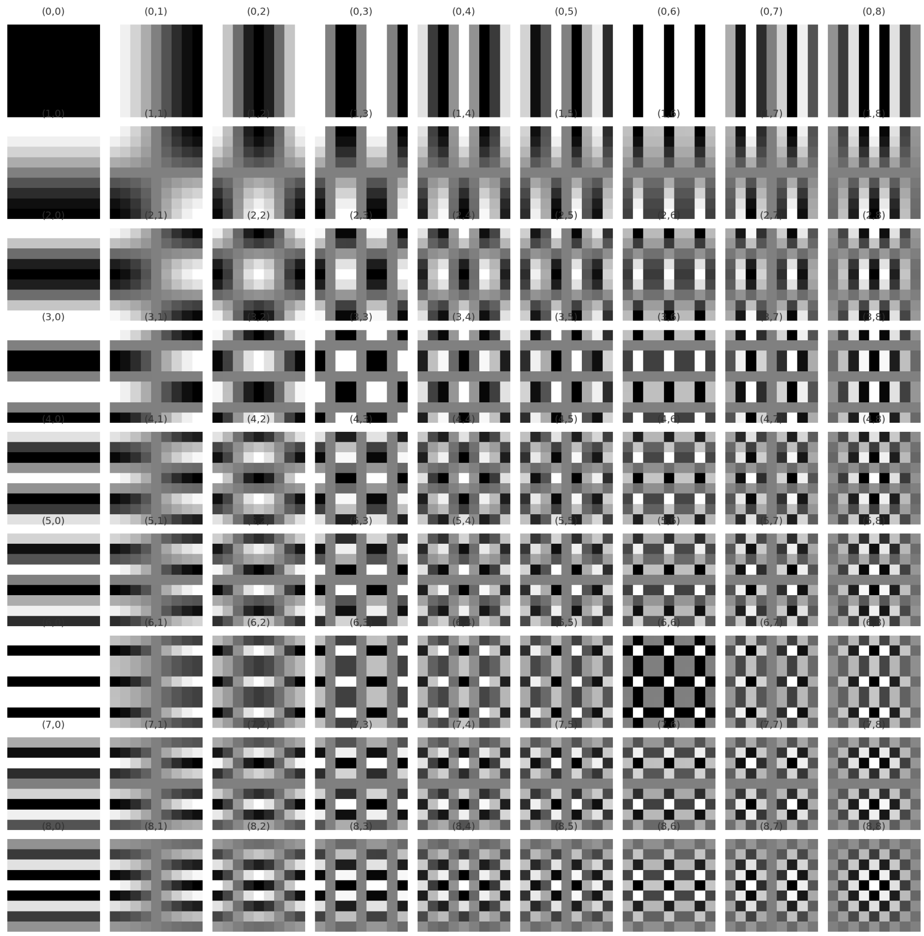

Just as with the DFT, the DCT gives us a way to express the original signal in terms of frequency content. However, because of the smoothness and boundary behavior of cosine functions, the DCT is particularly effective at energy compaction—most of the signal's energy tends to be captured in a few low-frequency coefficients $b_j$. This property underlies its widespread use in compression standards such as JPEG and MP3.

## 8.1 2D DCT for Images

The 1D DCT can be naturally extended to two dimensions, which is particularly useful in image processing, where signals are inherently two-dimensional. In the 2D case, the DCT basis functions are defined as:

$$\phi_{i,j}[k,l] = \cos\left(\pi \frac{i}{L}\left(k + \frac{1}{2}\right)\right) \cos\left(\pi \frac{j}{L}\left(\ell + \frac{1}{2}\right)\right),$$

where $i, j, k, \ell = 0, \ldots, L-1$, and each basis function $\phi_{i,j}$ corresponds to a pattern that oscillates $i/2$ times vertically and $j/2$ times horizontally across an $L \times L$ grid. An illustration of the full set of 2D DCT basis functions for $L = 9$ is shown below:




Given a grayscale image represented as a matrix $X \in \mathbb{R}^{L \times L}$, we can represent it as a linear combination of these 2D basis functions:

$$X[k, \ell] = \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} A[i, j]\, \phi_{i,j}[k, \ell],$$

where $A[i, j]$ are the DCT coefficients indicating how much of each basis pattern is present in the image.

Because natural images often contain predominantly low-frequency content, the DCT representation $Y$ tends to be sparse—most of the signal energy is concentrated in the top-left corner (low $u, v$). This property makes the DCT highly effective for image compression methods such as JPEG, where many high-frequency coefficients can be discarded with little perceptual loss.

## 8.2 Relation Between DCT and DFT

The DCT can be seen as a special case of the DFT applied to a symmetrically extended signal. Specifically, if we construct an even extension of a length-$L$ signal to a new signal of length $2L$, the DFT of this extended signal contains only real coefficients due to symmetry, and the cosine components in the DFT align with those of the DCT.

This relationship offers useful insights:

- The DCT avoids mixed cosine/sine components.
- It inherits orthogonality and invertibility from the DFT framework.

- Fast DCT algorithms are often derived by modifying FFT implementations on symmetric signals.

Because it uses only cosine components and has better boundary behavior, the DCT is often preferred over the DFT in applications like audio/image coding, where interpretability and energy compaction are key.