Stanford University

ENGR 76

Information Science and Engineering

Lecture 5: Frequency Domain Representation

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Course Announcements and Reminders

Reminders

- Mini-PSet 2
  - Practice problems and solutions
  - Andy's OH after class today
- Project 1a
  - Autograder for Huffman coding implementation

Recap

Shannon's Source Coding Theorem

Theorem

The entropy of a source equals the minimum number of bits per source symbol necessary on average to encode a sequence of **independent and identically distributed** symbols from that source. In general, this may require the use of block coding, where blocks of symbols are encoded together.

Dependent Sources

Dependent Sources

- Sequence of symbols $X_1, X_2, X_3 \ldots$
- For example, X_i and X_{i+1} are not independent
  - Knowing the value of X_i gives us some information about X_{i+1}
- Examples:
  - Text
  - Images
- Can we exploit this dependence to get better compression?

Block coding for INDEPENDENT sources

- Recall for independent and identically distributed (i.i.d.) sources:
- Applying Huffman code on blocks of size n

$$H(X_1, \ldots, X_n) \leq \bar{\ell}_{block,n} \leq H(X_1, \ldots, X_n) + 1$$

- i.i.d. source implies
  $H(X_1, \ldots, X_n) = H(X_1) + \ldots + H(X_n) = nH(X_1)$

$$nH(X_1) \leq \bar{\ell}_{block,n} \leq nH(X_1) + 1$$

- Average number of bits per symbol:

$$H(X_1) \leq \frac{\bar{\ell}_{block,n}}{n} \leq H(X_1) + \frac{1}{n}$$

What happens if not independent?

- Suppose $X_1, X_2, \dots$ are identically distributed but NOT independent
- Then $H(X_1, \dots, X_n) < nH(X_1)$
- Example: $H(X_1, \dots, X_n) = \frac{(n+1)}{2} H(X_1)$

What happens if not independent?

- Suppose $X_1, X_2, \ldots$ are identically distributed but NOT independent
- Then $H(X_1, \ldots, X_n) < nH(X_1)$
- Example: $H(X_1, \ldots, X_n) = \frac{(n+1)}{2} H(X_1)$:

$$
\begin{aligned}
& H(X_1, \ldots, X_n) \leq \bar{\ell}_{block,n} \leq H(X_1, \ldots, X_n) + 1 \\
\implies & \frac{(n+1)H(X_1)}{2} \leq \bar{\ell}_{block,n} \leq \frac{(n+1)H(X_1)}{2} + 1 \\
\implies & \frac{H(X_1)}{2} + \frac{H(X_1)}{2n} \leq \frac{\bar{\ell}_{block,n}}{n} \leq \frac{H(X_1)}{2} + \frac{H(X_1)}{2n} + \frac{1}{n}
\end{aligned}
$$

- Average bits per symbol for large n is now close to $H(X_1)/2$

Block Coding for Dependent Source

- Block coding for dependent sources can achieve average number of bits per symbol better than entropy
- Is this the only way to exploit dependence?

Example

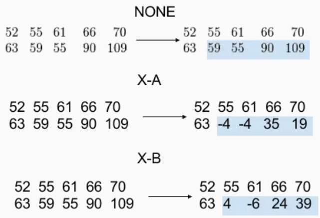
- Consider the sequence $X_1, X_2, \ldots$
- Special structure: $X_{i+1} = X_i + Z_i$
  - Z_i uniform random variable over $\{-1, 0, 1\}$
  - $P(Z_i = -1) = 1/3, P(Z_i = 0) = 1/3, P(Z_i = 1) = 1/3$
  - Z_i are independent with each other
- Example sequence: $480, 479, 478, 479, 479, 480, 480, 481, 480, \ldots$
- How would you store this sequence?

Example

- Given the sequence $X_1, X_2, X_3 \dots$
- Store the differences!
  - Store X_1
  - Then store consecutive differences: $X_2 - X_1, X_3 - X_2, X_4 - X_3, \dots$
  - These differences are now i.i.d.
- Example sequence: $480, 479, 478, 479, 479, 480, 480, 481, 480, \dots$
  - Store 480 separately
  - Consecutive differences: $-1, -1, 1, 0, 1, 0, 1, -1, \dots$ - encode this sequence

Example: Delta Coding

- **Delta Coding**
  - Storing data as the differences (*deltas*) between consecutive data points
  - Useful for *smooth data* like videos
- **PNG: image compression format**
  - First step is predicting next pixel based on left and above pixel
  - Example of delta coding[1]



[1] Figure from https://medium.com/@duhroach/how-png-works-f1174e3cc7b7

What did we effectively do?

- Represented original sequence in a form with *nice properties*

Representing Images

Cameraman

- Suppose you visit a website which has the following image

- But your internet is very slow
- How will it load?

Loading... (25%)

Loading... (50%)

Loading... (75%)

Loading... (100%)

How will it load?

- The image loaded block-by-block or pixel-by-pixel
- But it might load in another way...

Loading... (0.1%)

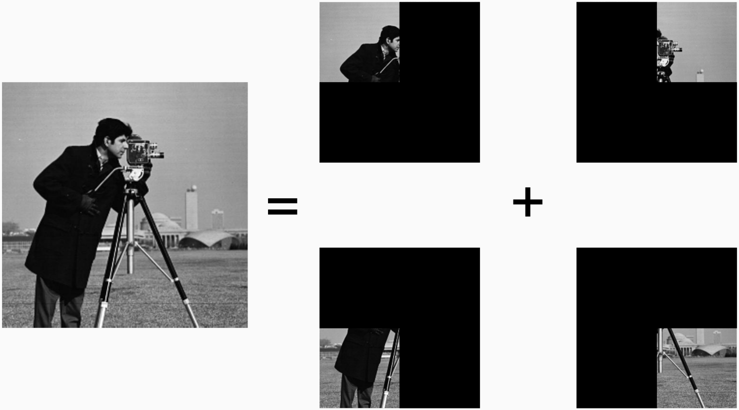
Loading... (1.6%)

Loading... (100%)

Representation

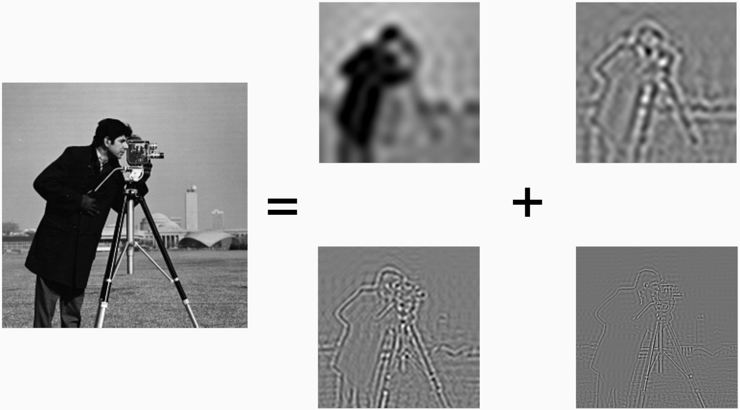
- Loaded from coarser to finer details
    - Image appears gradually, starting blurry and becoming sharper.
- How is this connected to representation?

Representation I: Pixel-by-pixel



- All four have equal importance (visual appearance)
- All four require the same amount of storage space in bits

Representation II



- The first three are much more important (visual appearance)
- The fourth requires 98% of storage space in bits (will show this later)

Why is Representation II useful?

- In this representation, most of the image's *visual information* can be stored using much fewer bits (e.g., 2% in previous example)
- Other advantages as well (next week)
- **Next Topic:** What is this representation?

Frequency Domain Representation

Signal

- Till now: information source is a sequence of random variables
- Different interpretation: information source is a **signal**
- Signal: a function that carries information
  - Notation: $x(\cdot)$
  - Can be a function of time (t) or space (m, n)

Signal

- Examples:
  - Electrical signal $x(t)$: representing voltage at each time t
  - Image $x[m, n]$: representing intensity of pixel at each pixel $[m, n]$
- Will focus on $x(t)$ for now
- Given a signal $x(t)$, how will you describe it?

How will you communicate $x(t)$?

Ideas:

1. Communicate the value of $x(t)$ at each t, i.e., at each time instant
2. Might be a known function, e.g., $x(t) = t^2 - t$
3. Is there a way to represent *any* signal using known functions?

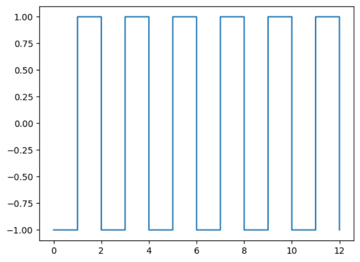
Periodic Signals

- Restrict attention to periodic signals

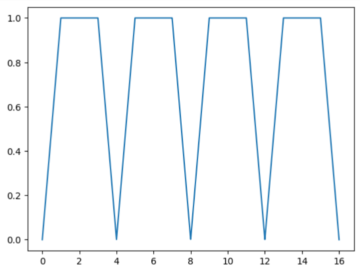
Definition

A signal $x(t)$ is periodic if there exists constant T such that $x(t+T) = x(t)$ for all t. The smallest $T > 0$ for which the signal satisfies this property is called the **period** of the signal.

Examples

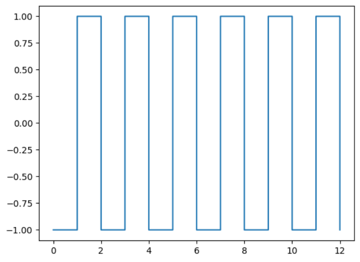


Period = ?

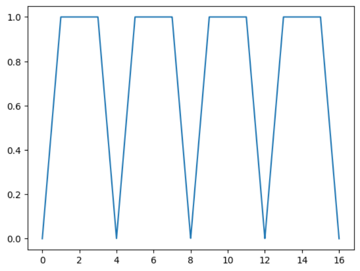


Period = ?

Examples

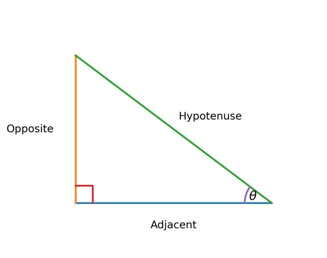


Period $= 2$ seconds



Period $= 4$ seconds

Sines and Cosines



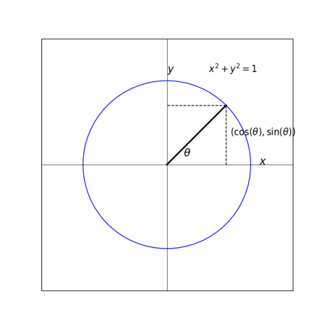
- Basic definition - using right angled triangle

$$\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$$

- θ in radians - this definition holds for $\theta \in (0, \frac{\pi}{2})$

Sines and Cosines



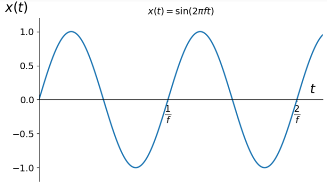
- $\cos(\theta)$ and $\sin(\theta)$ are x and y coordinates of a point on the unit circle that lies at angle θ radians from the x-axis
- $\sin(\theta + 2\pi k) = \sin(\theta)$ and $\cos(\theta + 2\pi k) = \cos(\theta)$ for integer k

Sine signal

- Let $\theta = 2\pi f t$:

$$x(t) = \sin(2\pi f t)$$

- t : time
- f : frequency or number of cycles per second



- What is the value of $x(t + 1/f)$?

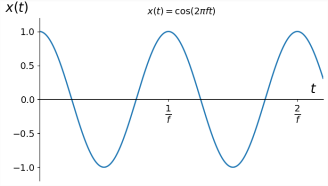
Sine signal

- What is the value of $x(t + 1/f)$?

$$
\begin{aligned}
x\left(t + \frac{1}{f}\right) &= \sin\left(2\pi f \left(t + \frac{1}{f}\right)\right) \\
&= \sin(2\pi f t + 2\pi) \\
&= \sin(2\pi f t) \\
&= x(t)
\end{aligned}
$$

- The signal $\sin(2\pi f t)$ is periodic with $T = 1/f$.

Cosine signal



- The signal $\cos(2\pi ft)$ is periodic with $T = 1/f$.

First Important Fact

Fact

Any *well-behaved*[2] periodic function with period T can be written in the following form:

$$x(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right).$$

[2] any reasonable function that we see in this course

What does this mean?

- Instead of describing a signal $x(t)$ which is periodic with period T in time domain by specifying the value at each time $t \in \mathbb{R}$, we can also describe it by specifying the amplitudes $b_0, \{a_j\}_{1 \leq i}$, and $\{b_j\}_{1 \leq j}$.
- **<u>Fourier</u> representation or frequency domain representation**

Fourier Analysis

- Joseph Fourier
  - Developed this theory to solve the heat equation - how heat flows through solids
- Many applications:
  - Fundamental to quantum mechanics
  - Image Processing (current discussion)
  - Communication systems (soon...)
  - Acoustics

Fourier Series Coefficients

- The coefficients $\{b_0, a_1, b_1, a_2, b_2, \ldots\}$
  - Unique for a signal $x(t)$
  - Need to know the period T and these coefficients to reconstruct $x(t)$
  - Names:
    - Fourier Series Coefficients
    - Frequency representation
    - Spectrum of the function

Thank You!