ENGR 76

Information Science and Engineering

Lecture 6: Frequency Domain Representation II

Siddharth Chandak

Representing using Sines and Cosines

- Rotating Angle Visualization (made using ChatGPT Canvas)

Representing using Sines and Cosines

Fact

Any *well-behaved*[1] periodic function with period T can be written in the following form:

$$x(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right).$$

[1] any reasonable function that we see in this course

Representing using Sines and Cosines

$$
\begin{aligned}
x(t) = b_0 & \\
& + a_1 \sin\left(2\pi \frac{1}{T} t\right) + b_1 \cos\left(2\pi \frac{1}{T} t\right) \\
& + a_2 \sin\left(2\pi \frac{2}{T} t\right) + b_2 \cos\left(2\pi \frac{2}{T} t\right) \\
& + a_3 \sin\left(2\pi \frac{3}{T} t\right) + b_3 \cos\left(2\pi \frac{3}{T} t\right) \\
& + \ldots
\end{aligned}
$$

- What are these terms and these functions?

The Constant Term

- Term b_0:
  - Corresponds to the zero frequency term
  - Average of the function
  - Shifts the function above or below by a constant
  - All other terms are sines and cosines which have zero average over a cycle

Harmonics

- The terms $\sin(2\pi \frac{j}{T} t)$ and $\cos(2\pi \frac{j}{T} t)$:
    - j^{th} harmonic
    - Frequency?
    - How many cycles does it complete in T seconds?

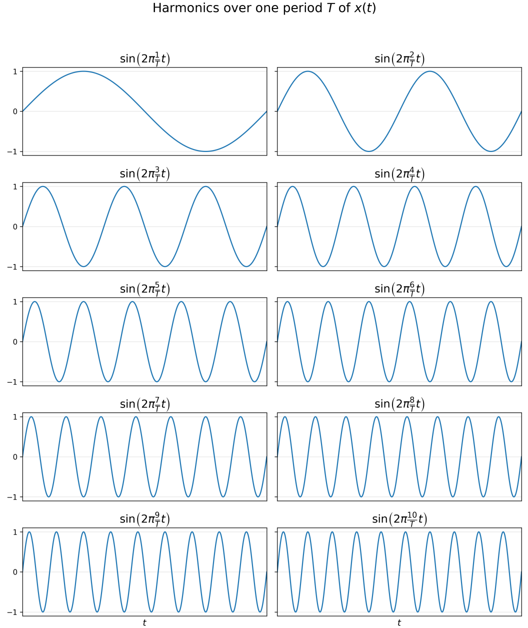
Harmonics

- The terms $\sin(2\pi \frac{j}{T} t)$ and $\cos(2\pi \frac{j}{T} t)$:
  - j^{th} harmonic
  - Frequency $f = \frac{j}{T}$
  - How many cycles does it complete in T seconds?
    - Number of cycles in 1 second is frequency $f = j/T$ cycles
    - Number of cycles in T seconds is $(j/T) \times T = j$ cycles

Harmonics

- All these harmonics are sines and cosines which complete integer number of cycles in T seconds, i.e., in one period of the signal $x(t)$
  - $\sin(2\pi \frac{1}{T} t)$ completes one cycle in one period of $x(t)$
  - $\sin(2\pi \frac{2}{T} t)$ completes two cycles in one period of $x(t)$
  - $\sin(2\pi \frac{100}{T} t)$ completes hundred cycles in one period of $x(t)$

Harmonics



Harmonics over one period T of x(t)
sin(2π 1/T t)
sin(2π 2/T t)
sin(2π 3/T t)
sin(2π 4/T t)
sin(2π 5/T t)
sin(2π 6/T t)
sin(2π 7/T t)
sin(2π 8/T t)
sin(2π 9/T t)
sin(2π 10/T t)
1
0
−1
t
t

Fourier Series Coefficients

- $\{b_0, a_1, b_2, a_2, b_2, \ldots\}$ are the Fourier series coefficients
  - a_j: how much $\sin(2\pi \frac{j}{T} t)$ contributes to the signal $x(t)$
  - Similar for b_j

Calculating these coefficients

There exist simple formulas to calculate the coefficients:

$$b_0 = \frac{1}{T}\int_0^T x(t)dt$$

$$a_j = \frac{2}{T}\int_0^T x(t)\sin\left(2\pi\frac{j}{T}t\right)dt$$

$$b_j = \frac{2}{T}\int_0^T x(t)\cos\left(2\pi\frac{j}{T}t\right)dt$$

Not important for this course!

Even Periodic Signals

Even and Odd Signals

Definition (Even signal)

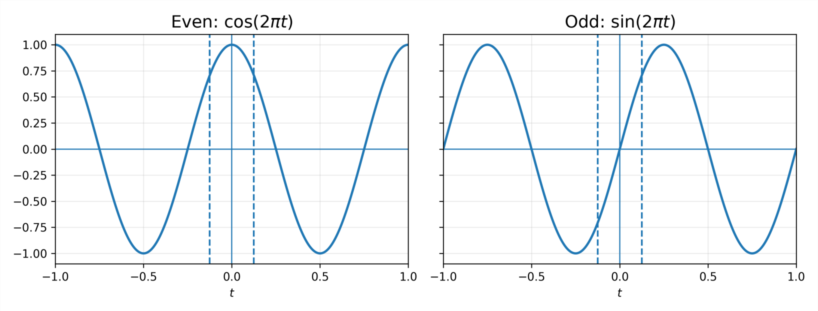
A signal $x(t)$ is even if $x(t) = x(-t)$ for all $t \in \mathbb{R}$, i.e., the signal is symmetric about the y-axis.

Definition (Odd Signal)

A signal $x(t)$ is odd if $x(t) = -x(-t)$ for all $t \in \mathbb{R}$.

- What about sine and cosine?

Even and Odd Signals



Sum of cosine and sine

Odd or even?

- Sum of two cosines of different frequencies?
- Sum of a sine signal and a cosine signal?

Sum of cosine and sine

- Sum of two cosines of different frequencies? **Even**
- Sum of a sine signal and a cosine signal? **Neither odd nor even**

Do we require sine terms in Fourier series representation of an even periodic signal?

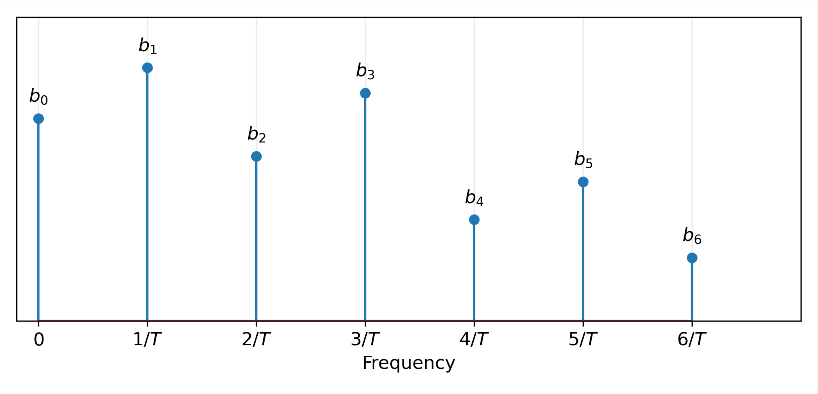
Fourier Cosine Series

Fact

Any even periodic function with period T can be written in the following form:

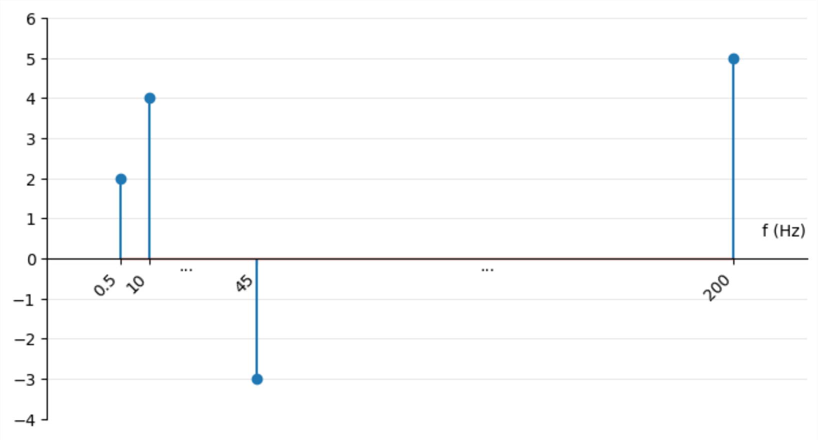
$$x(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos\left(2\pi \frac{j}{T} t\right).$$

Spectrum of a Signal

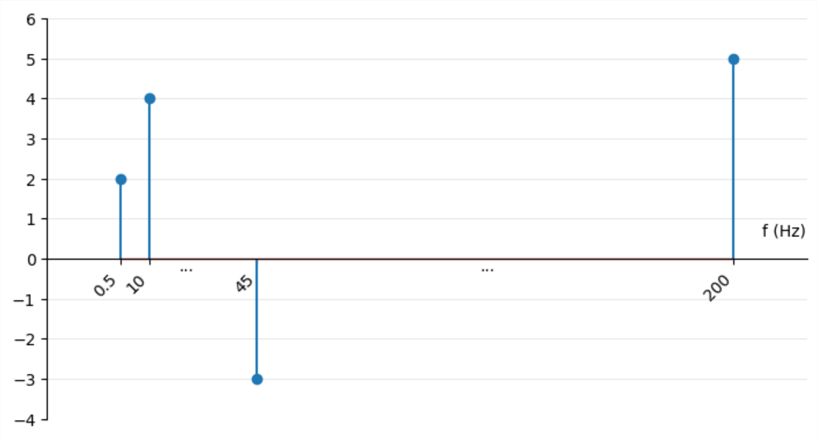


Spectrum of a Signal

- This stem plot can generally be for any arbitrary frequencies



Spectrum of a Signal



$$x(t) = 2\cos(2\pi 0.5t) + 4\cos(2\pi 10t) - 3\cos(2\pi 45t) + 5\cos(2\pi 200t)$$

Understanding Fourier Representation

$$x(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos(2\pi \frac{j}{T} t)$$

- What information about the signal do the lower frequency components contain?
- What information about the signal do the high frequency components contain?
- What happens if we approximate the signal using a finite summation, i.e., only take the first K harmonics: $\tilde{x}(t) = b_0 + \sum_{j=1}^{K} b_j \cos(2\pi \frac{j}{T} t)$?

Visualization

- Visualization for Fourier Cosine Series (again made using ChatGPT Canvas)

Lower and Higher frequency components

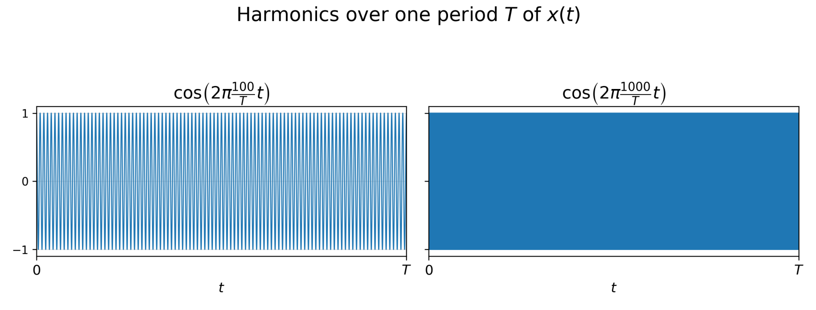
Lower frequency components:

- Capture the *coarse* or slowly varying structure of the signal
    - average of signal, envelopes, etc.
- Overall shape/trend

Higher frequency components:

- Capture the sharp changes or transitions
    - jumps, corners, etc.
    - jump discontinuities require more terms as compared to corners
- Local changes (and also noise)

High Frequency Harmonics



Low-pass filter

- Low-pass filter
  - Keeping only the low-frequency terms: discard high frequency terms by setting those coefficients to zero
  - Low-pass filter: keep b_j unchanged for frequency lower than cutoff and change b_j to zero for frequency greater than cutoff

Effects of a low-pass filter

- The signal gets smoothened
  - Edges are the first to lose fidelity
  - Sharp corners become rounded, sudden jumps are softened
- Reduces noise (example in Project 1b)
- Global structure is retained
- Implication: signal can be *well* approximated using a small number of terms
  - Direct intuitive relation with what we saw for images in last lecture

Side effect: Ringing

- Side effect: Ringing
  - Oscillations near sharp transitions
  - Called Gibbs Phenomenon

Side effect: Ringing

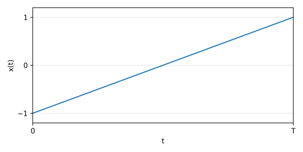
Finite-Duration Signals

Finite-duration Signals

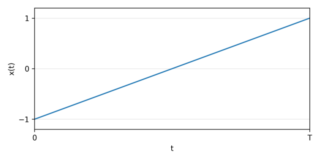
- Suppose we are given a signal of duration T seconds
- How to represent the signal using sines and cosines?

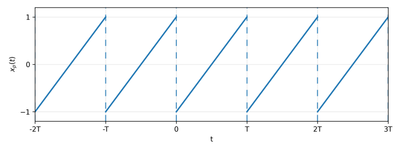
Periodic Extension

- Signal $x(t)$ defined from $t = 0$ to $t = T$
- **Periodic Extension:** $x_p(t)$ is obtained by repeating $x(t)$ again and again in $[T, 2T]$, $[2T, 3T]$, $[-T, 0]$, $[-2T, -T]$...



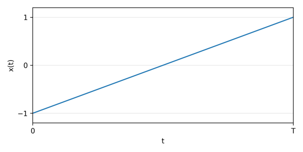
Periodic Extension



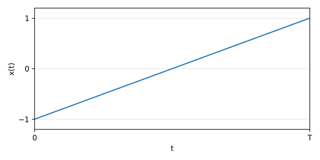


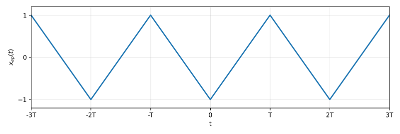
Even Periodic Extension

- Signal $x(t)$ defined from $t = 0$ to $t = T$
- **Even Periodic Extension** $x_{ep}(t)$:
  - First mirror $x(t)$ about y-axis to get its image from $-T$ to 0
  - Then repeat it again and again in $[T, 3T]$, $[-3T, -T]$...



Even Periodic Extension





Periodic and Even Periodic Extension

What are periods of $x_p(t)$ and $x_{ep}(t)$?

Periodic and Even Periodic Extension

Periods:

- Periodic extension $x_p(t)$: Period T
- Even periodic extension $x_{ep}(t)$: Period $2T$

Periodic and Even Periodic Extension

Which of these is desirable for Fourier representation?

Even Periodic Extensions

- If our original finite-duration signal is continuous in $[0, T]$, then its even periodic extension will also be continuous
- Periodic extension can have jump discontinuities
- Even periodic extension is desirable
  - Requires fewer terms in its Fourier representation

Fourier Series for Finite-Duration Signal

- Given a signal $x(t)$ defined in $[0, T]$
- Fourier Representation of $x(t)$ for $t \in [0, T]$:

$$x(t) = b_0 + \sum_{j=1}^{\infty} b_j \cos\left(2\pi \frac{j}{2T} t\right)$$

  - Note the $2T$ in $\frac{j}{2T}$: period of the even periodic extension is $2T$
  - Fourier representation is for even periodic extension but it coincides with the original signal in the finite duration $[0, T]$

Summary

- Represented periodic signals as sum of sines and cosines
  - Represented *even* periodic signals using cosines
- Studied what different frequency components capture
  - And how just a few terms can be used to approximate the signal
- Represented finite duration signals using cosines
- **Next Class and Project 1b**: Representing discrete-time signals!
  - Same intuition goes through!!

Thank You!