Stanford University

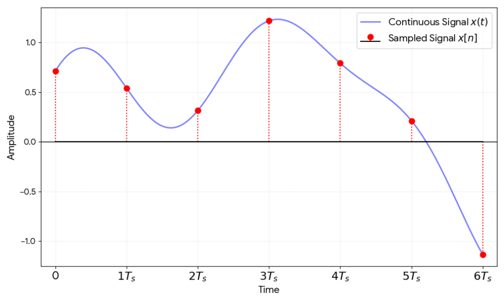
ENGR 76
Information Science and Engineering

Lecture 9: Sampling

Siddharth Chandak

Sampling

Sampling



- Given continuous time signal $x(t)$
  - Sampling it at uniform intervals of T_s seconds
  - Obtain discrete-time sequence $x(0), x(T_s), x(2T_s), \ldots$

Why sampling?

- Cannot store continuous-time signals using digital systems
- Need to store discrete samples
- Example: sampling an audio signal in time

Sampling Period

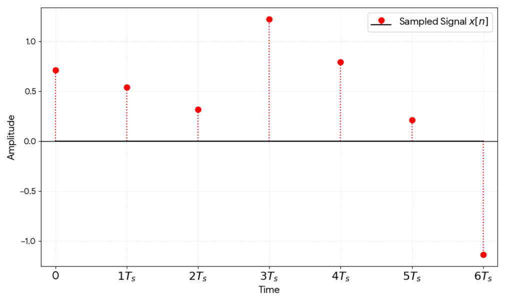
- $T_s > 0$:
  - Sampling period or sampling interval
  - One sample every T_s seconds
- $f_s = \frac{1}{T_s}$
  - Sampling frequency
  - f_s samples in one second

Reconstruction

- Given discrete samples $x(0), x(T_s), x(2T_s), \ldots$
- Reconstructing is the process of obtaining a continuous-time signal $\hat{x}(t)$ from these discrete samples
- Need for reconstruction:
  - Physical systems operate in continuous time
  - Example: speaker cannot play discrete samples
  - Needs a continuous-time audio signal
- Ideally, we would like **perfect reconstruction**:

$$\hat{x}(t) = x(t), \quad \text{for all } t$$

When is perfect reconstruction possible?



- We only observe the signal at discrete time instants:

$$\{x(0), x(T_s), x(2T_s), \ldots\}$$

- Between samples, the signal could in principle behave arbitrarily.
- How can a finite set of samples capture *all* the information in a continuous-time signal?

When is perfect reconstruction possible?

- Class of signals for which samples uniquely determine the signal?
- What condition does T_s (or f_s) need to satisfy?
- How to reconstruct $\hat{x}(t)$ from $\{x(0), x(T_s), x(2T_s), \ldots\}$ such that $\hat{x}(t) = x(t)$?

Bandlimited Signals

Bandlimited Signals

- Suppose $x(t)$ is periodic with period T (or equivalently is a signal of duration T)
- Recall the Fourier series representation:

$$x(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right)$$

Amplitude of the jth harmonic

$$x(t) = b_0 + \sum_{j=1}^{\infty} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right)$$

- Define $A_j = \sqrt{a_j^2 + b_j^2}$
- A_j
  - Strength of the j-th harmonic
  - Joint contribution of the sine and cosine term with frequency j/T
  - Useful for plotting the spectrum of the signal
  - Note that A_j is zero only if there is no contribution from the j-th harmonic (either sine or cosine)

Bandlimited Signal

Definition (Bandlimited Signal)

A signal is called bandlimited if its Fourier series representation contains only finitely many frequency components. Equivalently, A_j is non-zero only for finitely many values of j.

Examples

Bandlimited or not?

- $$x(t) = \sum_{j=1}^{\infty} \frac{1}{j} \sin\left(2\pi \frac{j}{T} t\right)$$
- $$x(t) = 5 + 2\sin\left(2\pi \frac{10}{T} t\right) - 9\sin\left(2\pi \frac{35}{T} t\right)$$
- Square wave?
- Triangle wave?

Examples

- Not bandlimited - Infinitely many components:

$$x(t) = \sum_{j=1}^{\infty} \frac{1}{j} \sin\left(2\pi \frac{j}{T} t\right)$$

- Bandlimited - Only three components:

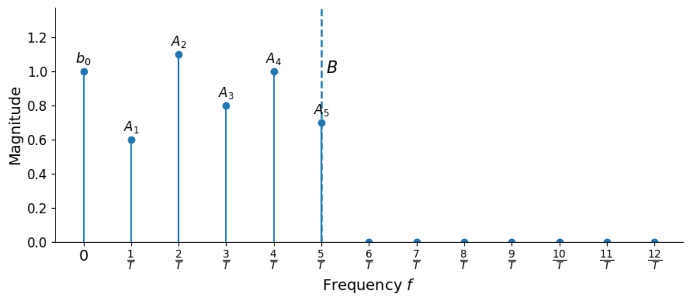
$$x(t) = 5 + 2\sin\left(2\pi \frac{10}{T} t\right) - 9\sin\left(2\pi \frac{35}{T} t\right)$$

- Square wave - Not bandlimited
  - Jump discontinuities - require infinity frequency components
- Triangle wave - Not bandlimited
  - Corners (not smooth) - require infinity frequency components

Baseband Signal

Definition (Baseband Signal)

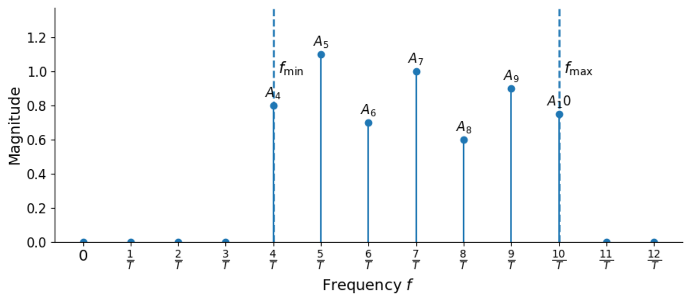
A signal is baseband if the Fourier series representation contain only those components which have frequencies in $[0, B]$, i.e. A_j is non-zero only if $\frac{j}{T} \leq B$ (or equivalently $A_j = 0$ if $j/T > B$).



Passband Signal

Definition (Passband Signal)

A signal is passband if the Fourier series representation contain only those components which have frequencies in $[f_{min}, f_{max}]$, i.e. A_j is non-zero only if $f_{min} \leq \frac{j}{T} \leq f_{max}$.



Band and Bandwidth

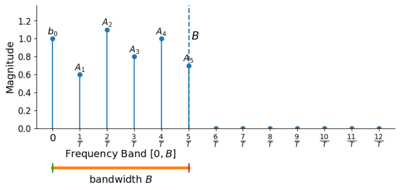
Definition (Frequency Band)

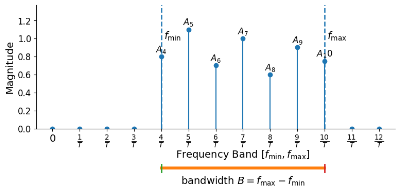
The frequency band of a signal is the smallest frequency interval that contains all frequencies present in the signal.

Definition (Bandwidth)

The bandwidth B of a signal is the width of the frequency band of a signal.

Baseband and Passband Signals





- Baseband Signal:
  - Frequency band: $[0, B]$ Hz
  - Bandwidth: B Hz
- Passband Signal:
  - Frequency band: $[f_{min}, f_{max}]$ Hz
  - Bandwidth: $B = f_{max} - f_{min}$ Hz

Examples

Frequency components? Frequency band? Baseband or Passband? Bandwidth?

- $x(t) = 15\cos(2\pi 25t) + \sin(2\pi 100t)$
- $x(t) = 12 - \cos(2\pi t) + 18\sin(2\pi 14t)$

Examples

- $x(t) = 15\cos(2\pi 25t) + \sin(2\pi 100t)$
    - Frequency components: 25 and 100 Hz.
    - Frequency band: $[25, 100]$ Hz.
    - Passband
    - Bandwidth $= 100 - 25 = 75$ Hz
- $x(t) = 12 - \cos(2\pi t) + 18\sin(2\pi 14t)$
    - Frequency components: $0, 1$ and 14 Hz.
    - Frequency band: $[0, 14]$ Hz.
    - Baseband
    - Bandwidth $= 14$ Hz
- What about $x(t) = 6 + \cos(2\pi t) - 9\sin(2\pi 14t)$?

Examples

- $x(t) = 12 - \cos(2\pi t) + 18\sin(2\pi 14t)$
- What about $x(t) = 6 + \cos(2\pi t) - 9\sin(2\pi 14t)$?
  - Same frequency components, frequency band, and bandwidth
  - Only the amplitudes of frequency components differ

Representing Bandlimited Signals

Representing Baseband Signals

- Consider $x(t)$ which has frequency band $[0, B]$ Hz

$$x(t) = b_0 + \sum_{j=1}^{???} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right)$$

- Where does the summation go up to?

Representing Baseband Signals

- Consider $x(t)$ which has frequency band $[0, B]$ Hz
- Only those components where frequency is in $[0, B]$

$$0 \leq \frac{j}{T} \leq B \implies 0 \leq j \leq BT$$

- The summation goes from $j = 0$ to $j = BT$

$$x(t) = b_0 + \sum_{j=1}^{BT} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right)$$

- Number of coefficients required to uniquely determine the signal?

Representing Baseband Signals

$$x(t) = b_0 + \sum_{j=1}^{BT} a_j \sin\left(2\pi \frac{j}{T} t\right) + b_j \cos\left(2\pi \frac{j}{T} t\right)$$

- Coefficients:

$$\{b_0, a_1, b_1, \ldots, a_{BT}, b_{BT}\}$$

- Number of coefficients required to uniquely determine the signal is $2BT + 1$
- How to calculate these coefficients?

Each sample gives a linear equation!

Consider example of $T = 1$ and $B = 2$

$$\begin{aligned} x(t) &= b_0 + \sum_{j=1}^{2} a_j \sin(2\pi j t) + b_j \cos(2\pi j t) \\ &= b_0 + a_1 \sin(2\pi t) + b_1 \cos(2\pi t) + a_2 \sin(4\pi t) + b_2 \cos(4\pi t) \end{aligned}$$

- At each sampling instance $t = 0, T_s, \ldots$?

Each sample gives a linear equation!

Consider example of $T = 1$ and $B = 2$

- At $t = 0$, observe $x(0)$.

$$\begin{aligned} x(0) &= b_0 + a_1 \sin(0) + b_1 \cos(0) + a_2 \sin(0) + b_2 \cos(0) \\ &= b_0 + b_1 + b_2 \end{aligned}$$

- At $t = T_s$, observe $x(T_s)$.

$$x(T_s) = b_0 + a_1 \sin(2\pi T_s) + b_1 \cos(2\pi T_s) + a_2 \sin(4\pi T_s) + b_2 \cos(4\pi T_s)$$

- At $t = 2T_s$, observe $x(2T_s)$

$$x(2T_s) = b_0 + a_1 \sin(4\pi T_s) + b_1 \cos(4\pi T_s) + a_2 \sin(8\pi T_s) + b_2 \cos(8\pi T_s)$$

Each sample gives a linear equation!

- The LHS in each of these equations is known (the observed samples $x(0), x(T_s), x(2T_s), \ldots$)
- The only unknowns are the coefficients $\{b_0, a_1, b_1, a_2, b_2\}$
- How many equations are needed to determine all unknowns (coefficients)?

Each sample gives a linear equation!

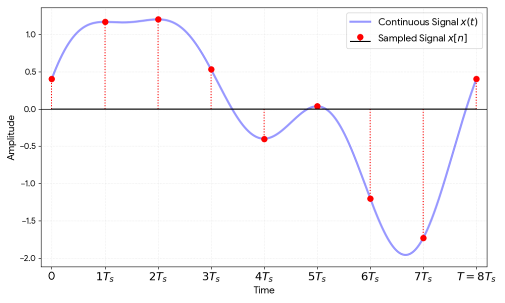
- The LHS in each of these equations is known (the observed samples $x(0), x(T_s), x(2T_s), \ldots$)
- The only unknowns are the coefficients $\{b_0, a_1, b_1, a_2, b_2\}$
- How many equations are needed to determine all unknowns (coefficients)?
    - For $T = 1$ and $B = 2$, we have 5 unknown coefficients, so we require 5 equations
    - Solving a system of 5 linear equations in 5 unknowns

Each sample gives a linear equation!

- The number of linear equations required to determine all the coefficients is equal to the number of coefficients
- Each sample gives a linear equation
- Number of coefficients is $2BT + 1$
  - Number of samples required is $2BT + 1$
- For signal of duration T and sampling period T_s, how many samples are obtained?

Number of samples

- Number of samples (for this finite-duration signal) is $\frac{T}{T_s} + 1$



Required Condition

- We require

$$\frac{T}{T_s} + 1 \geq 2BT + 1$$

- To avoid the critical case, we typically require strict inequality:

$$\frac{T}{T_s} + 1 > 2BT + 1$$
$$\implies T_s < \frac{1}{2B}$$

- Need sampling period to be smaller than $\frac{1}{2B}$
  - Require sampled points to be spaced less than $1/(2B)$ seconds apart
- Need sampling frequency larger than $2B$

Shannon-Nyquist Sampling Theorem

Theorem

Let $x(t)$ be a baseband signal with bandwidth B. Then $x(t)$ can be perfectly reconstructed from its samples $x(kT_s)$ for $k \in \mathbb{Z}$ if $T_s < 1/(2B)$ or $f_s > 2B$.

- $2B$ serves as a lower bound on sampling frequency required for *capturing all information*
- Can sample at higher frequency
  - Preferred in real systems

Significance

- Many real-world signals are approximately bandlimited
- Or they can be made bandlimited:
  - Audio signals
  - Human hearing is limited to 20 kHz: only need components till there
  - Common sampling frequency: 44.1 kHz
    - Safety margin
- For such signals, sampling enables
  - Lossless digital representation
  - Exact reconstruction

Why is this result surprising?

- A continuous-time signal has infinitely many values
- Sampling keeps only values at discrete time instants
- Between samples, the signal could *potentially* behave arbitrarily
- Still, under the right conditions, no information is lost

Intuition

- Band-limited signals are inherently *smooth* in time
- They cannot exhibit arbitrarily rapid changes
- Sampling fast enough captures all the degrees of freedom of the signal
- As a result, the samples uniquely determine the signal everywhere in time

Interpolation

How to reconstruct?

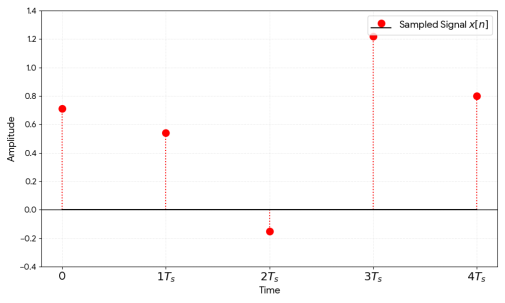
- Given samples $x(0), x(T_s), x(2T_s), \ldots$
- How to reconstruct the continuous time signal $\hat{x}(t)$?
- One possible solution:
    - Solve the set of equations discussed above
    - Obtain all coefficients
    - Gives us the continuous-time signal
    - Drawbacks?

How to reconstruct?

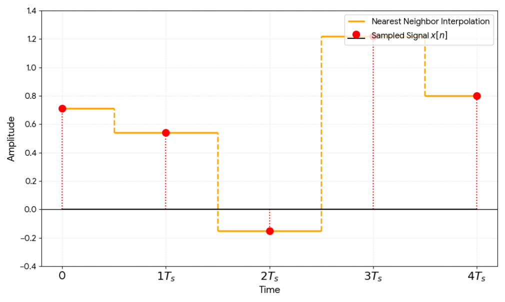
- Given samples $x(0), x(T_s), x(2T_s), \ldots$
- How to reconstruct the continuous time signal $\hat{x}(t)$?
- One possible solution:
    - Solve the set of equations discussed above
    - Obtain all coefficients
    - Gives us the continuous-time signal
    - Drawbacks:
        - Collect all samples till the end of the signal
        - Solve a huge system of linear equations
    - Not practical

Interpolation

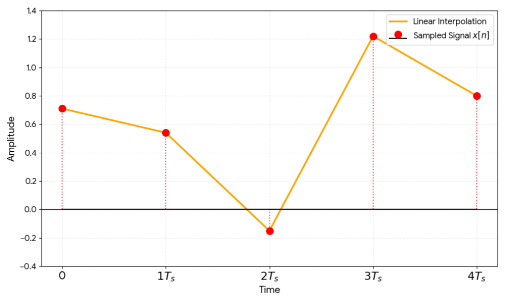
- Process of constructing continuous-time signal from discrete samples by filling in the values between the sampling instants



Nearest Neighbor Interpolation



Linear Interpolation



Basic Requirement

- Basic requirement:

$$\hat{x}(kT_s) = x(kT_s) \quad \text{for all } k \in \mathbb{Z}$$

  - We know the exact value of the signal at the sampling instances
  - The reconstructed signal should match at those instances
  - Both linear and nearest neighbor interpolation satisfy this

Perfect Reconstruction

- Suppose we have a bandlimited signal $x(t)$ with band $[0, B]$
- Samples $x(0), x(T_s), x(2T_s), \dots$
- Reconstructed signal using linear interpolation $\hat{x}_{lin}(t)$
- Reconstructed signal using nearest neighbor interpolation $\hat{x}_{nn}(t)$
- Can either of these be a perfect reconstruction?

Perfect Reconstruction

- Suppose we have a bandlimited signal $x(t)$ with band $[0, B]$
- Can either $\hat{x}_{nn}(t)$ or $\hat{x}_{lin}(t)$ be a perfect reconstruction?
  - No!
  - Nearest neighbor interpolation has jump discontinuities
  - Linear interpolation has corners
  - Both of these require infinite frequency components
  - But the original signal was bandlimited
- What interpolation gives perfect reconstruction?

Next Class

- Formalizing general interpolation functions
- Sinc interpolation: perfect reconstruction
- What happens if the assumption $f_s > 2B$ is not satisfied?

Thank You!