Constituency, Relations, and Functions

LINGUIST 130A/230A Section

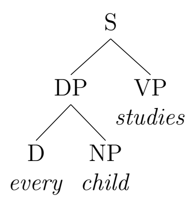
Winter 2022

1 Constituency

1.1 What is a constituent?

- Sentences have internal structure that is comprised of **constituents**.
- We have intuitions about what is and what is not a constituent in any sentence X.

(1)



The tree on the left claims that in the sentence *every child studies*, *every child* is a constituent, but *child studies* is not.

1.2 How can we identify constituents?

- There are **constituency tests** you can run by taking the string you want to test and creating a new sentence with it in different ways. If the resulting sentence is grammatical, that string is a constituent. If the resulting sentence is ungrammatical, that string is probably NOT a constituent[1].
- Examples of constituency tests:
  - Coordination test: Take the string and try to coordinate it in a new sentence.

    (2) If we want to test if *every child* is a constituent in *every child studies*: [Every child] and [many dogs] saw a bird.

    (3) If we want to test if *child studies* is a constituent in *every child studies*: *Every [child studies] and [man studies].

[1]Not all constituency tests work for all kinds of strings, so getting an ungrammatical sentence as a result of a constituency test doesn't necessarily mean that string is not a constituent. To work around this, it's always a good idea to run several types of constituency tests for every string you want to test.

⟹ [Every child] is a constituent, but [child studies] is (probably) NOT a constituent.

– Cleft test: Replace X in the following frame with the string you want to test and complete the rest of the new sentence: “It was **X** that . . . ”.

(4) If we want to test if *every child* is a constituent in *every child studies*:
It was [every child] that. . . left early.

(5) If we want to test if *child studies* is a constituent in *every child studies*:
*It was [child studies] that left early.

⟹ [Every child] is a constituent, but [child studies] is (probably) NOT a constituent.

– Question-answer test: Try to form a question that can be answered solely by the string you want to test.

(6) If we want to test if *every child* is a constituent in *every child studies*:
Q: Who hates waking up early?
A: [Every child].

(7) If we want to test if *child studies* is a constituent in *every child studies*:
Q: Who hates waking up early?
A: *[Child studies].

⟹ [Every child] is a constituent, but [child studies] is (probably) NOT a constituent.

2 Sets and ordering

- Order doesn't matter in sets. So, the set {a,b} is the same as the set {b,a}.
→ Curly brackets indicate that order doesn't matter; {a,b} = {b,a}

- Sometimes order matters. We'll use angled brackets to represent ordered pairs.
→ Angled brackets indicate that order matters; <a,b>≠< *b*,a>

  More generally, we can use angled brackets to represent *n*-tuples. An ordered triple has three elements <a, b, c>; an ordered *n*-tuple has *n*: <a_1, a_2,. . . , a_n>.

3 Relations

3.1 Definitions and examples

- Informally, a relation is something that holds or doesn't hold between two objects.

– E.g., a verb (predicate) like *loves* can be a relation: x and y are in the *loves* relation if x loves y. This obviously isn't the same as saying that y loves x, so we need ordered pairs.

- More formally:

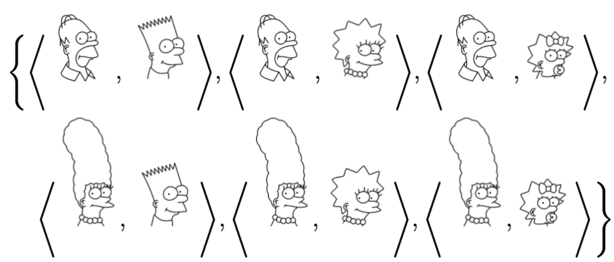
Definition 3.1. A **relation** is a set of tuples of the same lengths. The length n is called the **arity** of the relation. A 2-ary relation is also called a **binary** relation, and a 3-ary relation is also called a **ternary** relation.

Example 3.2. Some familiar binary relations: $=$ (*equal to*), $<$ (*less than*), $>$ (*more than*).

Example 3.3. *Less than* relation in a set of prime numbers below 10 (i.e., 2, 3, 5, 7):

$\{<2,3>,<2,5>,<2,7>,<3,5>,<3,7>,<5,7>\}$

Example 3.4. *The parent of* relation in the set of Simpson family members is defined by the following set:



We could also write this as:

$\{\langle x, y\rangle : x \textit{ is a parent of } y \text{ and } x, y \text{ are in the Simpsons family}\}$

- **Infix notation** for a binary relation R: xRy, which means just the same as $\langle x, y\rangle \in R$. (This is how we typically use binary relations such as $=$, $<$ and $>$.)

Example 3.5. We can treat transitive verbs such as *give* as denoting ternary relations:

$$[\![\text{give}]\!] = \{\langle x, y, z\rangle \mid x \text{ gives } y \text{ to } z\}$$

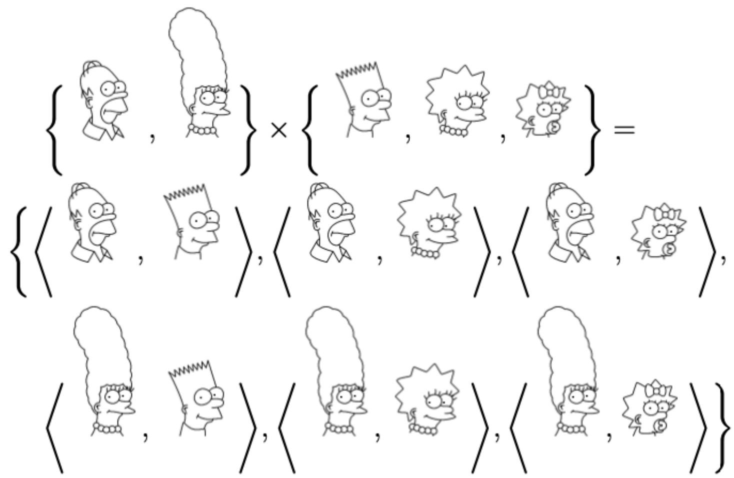
– Note: This is not the best analysis of transitive verbs (we will discuss better alternatives later), but it does capture a core aspect of their meanings, e.g., in the sentence *John gives the book to Mary*, the subject, the direct object, and the indirect object must be in a specific relation in order for it to be true.

Definition 3.6. The **Cartesian product** of sets A_1, A_2,..., A_n, written as $A_1 \times A_2, \ldots, \times A_n$, is a set of n-tuples defined as follows.

$$A_1 \times A_2, \ldots, \times A_n = \{\langle a_1, a_2, \ldots, a_n\rangle | a_1 \in A_1 \text{ and } a_2 \in A_2, \ldots, \text{ and } a_n \in A_n\}$$

If $A_1 = A_2 = \ldots = A_n$, we can also write their Cartesian product as A^n. In this case, we will also call any relation R that is a subset of A^n an n-ary relation on A.

Example 3.7. The Cartesian product of parent Simpsons with child Simpsons gives us the *is a parent of* relation on the Simpsons family:



Definition 3.8. For a binary relation R, its **inverse** relation, written as R^{-1}, is a relation defined as follows:

$$R^{-1} = \{\langle x, y\rangle | \langle y, x\rangle \in R\}$$

Definition 3.9. For a binary relation R, its **domain** and **range** are sets defined as follows.

$$\text{Domain}(R) = \{x \mid \text{there is some } y \text{ such that } \langle x, y\rangle \in R\}$$
$$\text{Range}(R) = \{x \mid \text{there is some } y \text{ such that } \langle y, x\rangle \in R\}$$

- That is, given a relation $R = A \times B$, the set of first coordinates A is the **domain** of R, and the set of second coordinates B is the **range** of R.

3.2 Properties of relations

Definition 3.10. A relation R is **reflexive** iff for all x, $\langle x, x\rangle \in R$.
A relation R is **irreflexive** iff for all x, $\langle x, x\rangle \notin R$.

Example 3.11. Equality is a reflexive relation; for any x, $x = x$.

Definition 3.12. A relation R is **symmetric** iff for all x, y if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \in R$.
A relation R is **anti-symmetric** iff for all distinct x and y (i.e., $x \neq y$), if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \notin R$.
A relation R is **asymmetric** iff for all x, y (which may or may not be the same), if $\langle x, y\rangle \in R$, then $\langle y, x\rangle \notin R$.

Example 3.13. The relation *sibling of* is symmetric, it work both ways.

Definition 3.14. A relation R is **transitive** iff for all x, y, z, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \in R$.
A relation R is **anti-transitive** iff for all x, y, z, if $\langle x, y\rangle \in R$ and $\langle y, z\rangle \in R$, then $\langle x, z\rangle \notin R$.

Example 3.15. *Less than* is a transitive relation: if $x < y$ and $y < z$, we have $x < z$.

4 Functions

4.1 Definition and examples

- Intuitively, a **function** from A to B is a machine that takes an object x as input and outputs another object y. We will call this input-output relation a function if such a relation is deterministic, i.e., for any input x there is at most one output y (it is OK if the function does not output anything at all for x, in which case we will say the function is undefined for x). The input of a function is also called its **argument**, and the output of a function is also called its **value**.

- We write $f : A \to B$, which means that f is a function that takes elements of the set A to elements of the set B. A is the **domain**, and B is the **range** (sometimes called the **co-domain**).

- More formally, a function is a relation that satisfies an additional requirement.

  Definition 4.1. A relation f is a **function** iff for every x, there is at most one y such that $\langle x, y\rangle \in f$.

- For a function f, we typically write $f(x) = y$ or $y = f(x)$ instead of $\langle x, y\rangle \in f$.

  Example 4.2. The relation *the next natural number of* is a function. It is called the **successor function**, and written as S. For example, $S(0) = 1$, $S(2) = 3$, and $S(100) = 101$.

  Example 4.3. The inverse of the successor function, S^{-1}, is the relation *the natural number right before*, which is also a function. For example, $S^{-1}(1) = 0$, $S^{-1}(3) = 2$, $S^{-1}(101) = 100$.

- The inverse of a function is by definition always a relation. However, it is not necessarily a function, e.g., *the height of* is a function, but its inverse is not.

- Since a function is a relation, we can specify it by listing all the pairs in the set. To highlight the directionality of the input-output relation, we often write $x \mapsto y$ instead of $\langle x, y\rangle$ when specifying a function.

  Example 4.4. The function *the suit name of* can be specified as follows:

$$\clubsuit \mapsto \text{club}, \diamondsuit \mapsto \text{diamond}, \heartsuit \mapsto \text{heart}, \spadesuit \mapsto \text{spade}$$

4.2 Properties of functions

Definition 4.5. A function f is **total** on a set A iff for every $x \in A$, there is a y such that $\langle x, y \rangle \in f$. Otherwise it is **partial**.

Example 4.6. Let $\mathbb{N}$ be the set of natural numbers. The successor function S is total on $\mathbb{N}$. In contrast, the inverse of the successor function, S^{-1}, is not total on $\mathbb{N}$, because $S^{-1}(0)$ is undefined, i.e., there is no y such that $\langle 0, y \rangle \in S^{-1}$.

Definition 4.7. A function $f : A \to B$ is **surjective** (or **onto**) iff $\text{Range}(f) = B$

Example 4.8. Let $A = \mathbb{Z}$ (the integers) and $B = 2\mathbb{Z}$ (the even integers). Then, $f : A \to B$ defined by $f(a) = 2a$ is onto since every even integer is a multiple by 2 of some integer.

Example 4.9. The successor function $S : \mathbb{N} \to \mathbb{N}$ is not surjective/onto because its range does not include 0. In contrast, its inverse, $S^{-1} : \mathbb{N} \to \mathbb{N}$, is surjective/onto.

Definition 4.10. A function $f : A \to B$ is **injective** (or **one-to-one**) iff for any $y \in B$, there is at most one x such that $f(x) = y$.

Example 4.11. Let A and B both be $\mathbb{Z}$, the integers, and let $f : A \to B$ be defined by $f(a) = a + 2$. Then f is one-to-one. Any $b \in B$ is uniquely mapped to by $b2 \in A : f(b2) = (b2) + 2 = b$.

Example 4.12. The successor function $S : \mathbb{N} \to \mathbb{N}$ is injective/one-to-one, and so is its inverse, $S^{-1} : \mathbb{N} \to \mathbb{N}$. The square function ^{2}: $\mathbb{R} \to \mathbb{R}$ R is not injective/one-to-one because, e.g., $4 = 2^2 = (2)^2$.

Definition 4.13. A function $f : A \to B$ is **bijective** (or a **one-to-one correspondence**) iff it is total on A, injective/one-to-one and surjective/onto.

Example 4.14. The successor function $S : \mathbb{N} \to \mathbb{N}$ is not bijective because it is not surjective/onto. Its inverse, S^{-1} is not bijective either, because it is not total on $\mathbb{N}$. The square function ^{2}: $\mathbb{R} \to \mathbb{R}$ is not bijective because it is not injective/one-to-one. The cube function ^{3}: $\mathbb{R} \to \mathbb{R}$ is bijective, because it is total on $\mathbb{R}$, injective/one-to-one, and surjective/onto. Also, the **identity function** $\text{id} : A \mapsto A$, which always simply returns the input (i.e., $\text{id}(x) = x$ for any $x \in A$), is trivially a bijection.

4.3 Truth values and characteristic functions

- There are two **truth values**: true and false. We often use T and F (or 1 and 0) to represent them.
- The set containing the two truth values is called the **Boolean domain**, and written as $\mathbb{B}$.

  Definition 4.15. $\mathbb{B} = \{\text{T, F}\}$

- Suppose we have a total function $f : D \to B$. Assume the domain D is the set of suits and f is defined as follows:

  Example 4.16. Let D be the set $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, and

  $$f = \{\clubsuit \mapsto \mathrm{F}, \diamondsuit \mapsto \mathrm{T}, \heartsuit \mapsto \mathrm{T}, \spadesuit \mapsto \mathrm{F}\}$$

  f returns T iff the input is a red suit. Since there are only two possible output values (T and F), once we know the set of inputs that the function will output T (call it A), we can determine the output of the function for any input (i.e., if the input is in A, then the function will output T, otherwise it will output F); the set A encodes all the relevant information we need to determine the output of the function for any input. We call this set A the **characteristic set** of the function f.

  Definition 4.17. For a total function $f : D \mapsto \mathbb{B}$, its **characteristic set** is defined to be the set $\{x | f(x) = \mathrm{T}\}$.

  Example 4.18. The characteristic set of the function in the previous example is $\{\diamondsuit, \heartsuit\}$.

- We have seen above how we can use sets to represent the relevant information of a function. We can also do it the other way around, i.e., use functions to represent the relevant information of a set.

  Definition 4.19. For a domain/universe D and a subset A, the **characteristic function** of A is the function $f : D \to \mathbb{B}$ that satisfies the following requirement:
  $f(x) = \mathrm{T}$ iff xA.

  Example 4.20. Suppose the domain/universe D is the set $\{\clubsuit, \diamondsuit, \heartsuit, \spadesuit\}$, then the characteristic function of $\{\diamondsuit, \heartsuit\}$ is the function specified in example 4.16.