Esoteric logic, concerned? No.

"No entity without identity."
Aim: find the correct identity criteria for proofs

Syntax of derivations can introduce spurious distinctions.
— Only to their syntactic representations as derivations in some proof system

Proofs: we have no direct access to proofs.
— Instead of "when is a truth?" ask "what is a proof of A?"

Proofs as the sense of formulas:

Proofs as First-Class Objects

Curry-Howard Isomorphisms

What You Need to Remember

Basic of proof theory

1. Curry-Howard Isomorphism

Motivated by proof theoretic considerations
Mechanism by which give derivations assemble meaningful terms
Proofs as first-class objects

2. Linear logic

Motivated by proof theoretic considerations
Mechanism by which give derivations assemble meaningful terms
Proofs as first-class objects

Ling 233B 1/16/02

2
(a) works for linear implication. \( \rightarrow \)

\[
\begin{align*}
\vdash & f \\
\vdash & g \\
\vdash & b \\
\vdash & a \\
\vdash & a \\
\end{align*}
\]

\( A \leftrightarrow B \): function \( f \) that takes a proof \( a \) to give a proof \( f(a) \) of \( B \)

\[
\begin{align*}
\vdash & f \\
\vdash & b \\
\vdash & a \\
\vdash & a \\
\end{align*}
\]

Curry-Howard Isomorphism

Natural deduction rule for (intuitionistic) implication elimination:

---

**Implication Elimination as Functional Application**

**Semantics of Proofs**

---

Distinct proofs parses have distinct meanings

---

**Parse Trees as Proof Trees**

---

Corresponds to two distinct proofs of:
Syntactically-defined evaluation correspondence to some proof

Definition: Intensional Identity Criteria for Proofs

Normalization of proofs: Isomorphic to -Reduction of proof terms

Intensive Relation between Logic and Type-Theory.

But it doesn't work for all logical or proof systems

\( \chi = \text{pairing of proof rules with } \forall \text{-operations on proof terms} \)

 Curry-Howard Isomorphism (CHI)

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\[ \text{Theare are not really distict proofs:} \]

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**Remove detours by \(//\)-normalization rules:**

**Normalization:**

\[ \frac{\forall \psi = \forall \psi}{\psi} \]

\[ \frac{\exists \psi = \exists \psi}{\psi} \]

\[ \frac{\exists \psi = \exists \psi}{\psi} \]

\[ \frac{\exists \psi = \exists \psi}{\psi} \]

\[ \frac{\exists \psi = \exists \psi}{\psi} \]

**Two types of detour:**

<table>
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<th>4</th>
<th>13</th>
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</table>

**Some Normalization Rules**

- Introduction of a connective immediately followed by its elimination
- Elimination of a connective immediately followed by its introduction

**Detours in Proofs**

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**Proof Normalization**

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</table>

**Introduction**

Each connective defined by paired introduction & elimination rules
Can't cover all of this here.

Other logics can be coded in linear logic
Show how this encodes interesting Identity criteria
Remove structural rules to get linear logic
Step back from natural deduction to sequent calculus
Method:

More general Identity criteria for proofs:

Curry-Howard gives non-trivial Identity criteria for proofs, but:

Curry-Howard and Proof Identity

Limitations of Curry-Howard
Different sequent-derivations of one ND proof

\[ \frac{A \lor B}{A} \]

\[ \frac{A \lor B}{B} \]

Both the sequent derivations correspond to the same natural deduction proof

Choose main connective to split on

Work backwards from conclusion

How to do proofs in sequent calculus

Natural deduction for \( \land \) \& \( \lor \)
Adjoint and multiplicative connectives
Removing structural rules

Sequential calculus

Alternative proof system: sequential version of natural deduction

Linear Logic: Overview
Permutation: order of premises remains immaterial

No duplication or discharger: premises become resources.

- multisets allow permutation of premises only
- follows from the multisets \( \phi_1 \ldots \phi_n \)

In linear logic:

\[
\phi_1 \ldots \phi_n \vdash \phi
\]

In traditional logic:

\[\phi_1 \ldots \phi_n \vdash \phi\]

Contraction & Weakening: Premises as Resources

<table>
<thead>
<tr>
<th>Rule</th>
<th>Formulas</th>
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<tbody>
<tr>
<td>Contraction</td>
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<td>Weakening</td>
<td>( L \vdash A )</td>
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<tr>
<td>Weakening</td>
<td>( L \vdash A )</td>
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Consequences of Removing Structural Rules

If we remove them:

Structural rules implicit in traditional logic.

And we get linear logic:

- We get only distinct versions of each familiar connective.
- We tighten identity criteria for proofs.
- We prevent reduction of premises and multiple discharge of assumptions.
- Derivations are from multisets of premises.

\[\phi \vdash \phi_1 \ldots \phi_n \]
Additive conjunction (with)

Premise $a$ and $b$ bundled together

Let $a$ and $b$ bundled to prove $a \otimes b$.

Without contraction & weakening, rules define alternate connectives

\[
\begin{align*}
&\text{Additive and Multiplicative Conjunction} \\
&\text{Multiplicative and Additive Conjunction}
\end{align*}
\]

\[
\begin{align*}
&\text{Similarly for } \forall^c\text{.}
\end{align*}
\]

Rules $\forall^a$ and $\forall^c$, interchangeable given contraction and weakening:

Where $\forall^a$ is a special case of $\forall^c$, $\forall_i^c = \forall_i^a$.

\[
\begin{align*}
&\text{(Multiplicative)} \\
&\text{Multiplicative:}
\end{align*}
\]

\[
\begin{align*}
&\text{Additive:}
\end{align*}
\]

\[
\begin{align*}
&\text{Universal}
\end{align*}
\]

Interchangeability given Contraction & Weakening

Alternate Rule for Conjunction ($\forall^c$)
Can discharge non-existential assumptions:

**Non-linear only**

\[
\frac{x \vdash (\forall a \ x a) \in \text{env}}{\exists x \vdash (\exists a \ x a) \in \text{env}}
\]

Can discharge multiple (co-indexed) assumptions at once:

**Non-linear only**

\[
\frac{x \vdash C \in \text{env} \land (a \vdash C) \in \text{env}}{\exists x \vdash C \land (a \vdash C) \in \text{env}}
\]

Needn't discharge assumptions in order they are introduced

**Premisses are just undischarged assumptions**

---

Linear and Non-linear Assumptions

---

**Pragmata** (Natural Deduction for Multiplicative) Linear Logic

---

A Pile of Connectives
$\vdash \forall : (x) d \forall : x$

$\vdash (x) d$

$\vdash [V : x]$

Using normalization of $\eta$-reduction, show the essential equivalence of

Curry-Howard

Linear Logic

Exercise

What You Need to Remember