Math 108 Homework 1 Solutions

Problem 1B
Suppose that G is a simple graph on 10 vertices that is not connected. Prove that G has at most 36 eges. Can equality occur?

Proof. Let v be one of the vertices of G. Let A be the connected component of G containing v, and let B be the remainder of G, so that $B = G \setminus A$. Let a be the number of vertices in A, and b the number of vertices in B. Since G is not connected, both A and B are nonempty, so $a, b \geq 1$, and since G has ten vertices, $b = 10 - a$. By the definition of a connected component, there are no edges in G between vertices in A and vertices in B, so that the number of edges in G is bounded above by sum of the numbers of edges in the complete graphs on the vertices of A and of B.

The complete graph with n nodes has $n(n-1)/2$ edges, so that the number of edges in G is bounded above by

$$\begin{aligned}
& \max_{a \in \{1,2,\ldots,9\}} a(a-1)/2 + (10-a)(10-a-1)/2 \\
= & \max_{a \in \{1,2,\ldots,9\}} \frac{a^2 - a + 100 - 20a + a^2 - 10 + a}{2} \\
= & \max_{a \in \{1,2,\ldots,9\}} a^2 - 10a + 45
\end{aligned}$$

It is readily verified that this maximum occurs for $a = 1$ (or $a = 9$), and takes the value 36. Thus G has at most 36 edges. Moreover, equality can occur, if G is the union of the complete graph on 9 vertices with an extra lone vertex not adjacent to any other. □

Problem 1C
Show that a connected graph on n vertices is a tree if and only if it has $n - 1$ edges.

Proof.
($\Rightarrow$) Suppose that G is a tree with n vertices. Let v_1 be a given vertex in G. For $1 \leq m < n$, let $V_m = \{v_1, ..., v_m\}$, and let v_{m+1} be a vertex in G which is not contained in V_m but which is adjacent to one of the vertices in V_m. Also, let G_m be the induced subgraph of G on the vertex set V_m. (Note that the manner in which v_{m+1} is selected implies that G_m is connected for all m.) Then I claim that for each $1 \leq m \leq n$, G_m has $m - 1$ edges.

I prove this by induction. The claim is trivial for $m = 1$. Suppose the claim holds for $m = k$, i.e. that G_k has $k - 1$ edges. Then, since $G_{k+1} \supseteq G_k$, and G_{k+1} is connected,

G_{k+1} contains all of the $k-1$ edges that G_k contains, and also contains at least one edge incident to v_{k+1}, so that G_{k+1} contains at least k edges. Suppose that G_{k+1} contains more than one edge incident to v_{k+1}. Thus, G_{k+1} contains the edges (v_{k+1}, v_a) and (v_{k+1}, v_b) for some $a \neq b$, $a, b \leq k$. Since G_k is connected, there is a path from v_a to v_b in G_k, and appending the edges (v_b, v_{k+1}) and (v_{k+1}, v_a) closes a loop in G_{k+1}. Since $G_{k+1} \subseteq G$, this shows that G is not a tree, a contradiction. Thus, G_{k+1} has precisely k edges. Since this holds for $G_n = G$, we may conclude that if G is a tree with n vertices, then G has $n-1$ edges.

($\Leftarrow$) Suppose that G is a connected graph with n vertices which is not a tree. Let the v_m, V_m and G_m be as defined in the first direction of the proof. The number of edges in G_1 is zero, and by similar arguments to those given above, the number of edges in G_{m+1} is at least one greater than the number of edges G_m. Let M be the smallest positive integer such that G_M contains a loop. Then v_M must be one of the vertices involved in the loop, and as a result, must be adjacent to at least two of the vertices in V_{M-1}. As a result, the number of edges in G_M is at least two greater than the number of edges in G_{M-1}. Thus, the number of edges in $G_n = G$ is at least $n-2+2 = n > n-1$. Thus, if G is a connected graph with n vertices which is not a tree, G does not have $n-1$ edges. □

Problem 1G

Show that a finite simple graph with more than one vertex has at least two vertices with the same degree.

Proof.

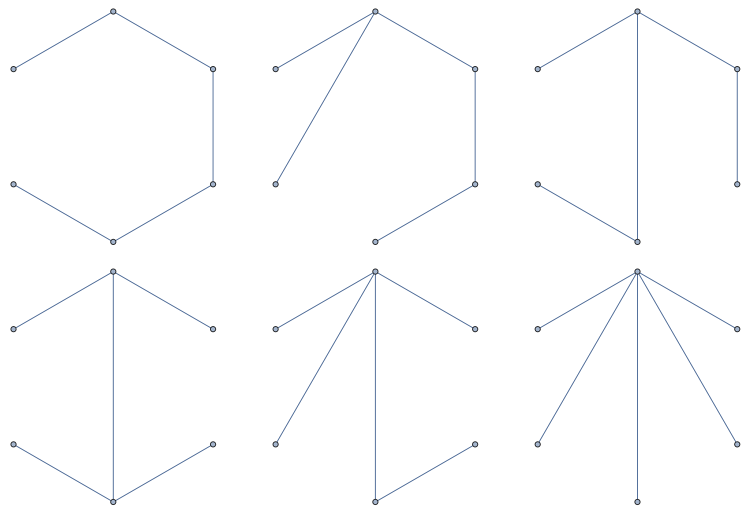
First, suppose that G is a connected finite simple graph with n vertices. Then every vertex in G has degree between 1 and $n-1$ (the degree of a given vertex cannot be zero since G is connected, and is at most $n-1$ since G is simple). Since there are n vertices in G with degree between 1 and $n-1$, the pigeon hole principle lets us conclude that there is some integer k between 1 and $n-1$ such that two or more vertices have degree k.

Now, suppose G is an arbitrary finite simple graph (not necessarily connected). If G has any connected component consisting of two or more vertices, the above argument shows that that component contains two vertices with the same degree, and therefore G does as well. On the other hand, if G has no connected components with more than one vertex, then every vertex in G has degree zero, and so there are multiple vertices in G with the same degree. □

Problem 2A

Find the six nonisomorphic trees on 6 vertices, and for each compute the number of distinct spanning trees in K_6 isomorphic to it.

See Figure 1 for the six isomorphism classes. These were obtained by, for each $k = 2, 3, 4, 5$, assuming that k was the highest degree of a vertex in the graph. For $k \neq 3$, there is precisely one connected tree on 6 vertices such that the largest degree is k, and for $k = 3$, there are the three possibilities depicted.



Proof.

Figure 1: The six isomorphism classes of simple graphs on six vertices.

As noted in the text, the number of distinct spanning trees isomorphic to a given one of these graphs is equal to 6! divided by the size of the automorphism group of the graph. Proceeding clockwise from the top left graph, we may compute the order of the automorphism groups as follows:

1. Any automorphism of the path graphs must send each endpoint (degree 1 vertex) to an endpoint, and there are only two maps that do that: the identity, and the map which reverses the path. Thus, this graph has automorphism group of order 2.

2. The top middle graph has precisely one vertex of degree 3, and so this vertex must be fixed under any automorphism. In addition, the degree one vertices adjacent to the degree three vertex are closed under the automorphism group, so that there are only two automorphisms: the identity, and the map which switches these two vertices (no other map preserves the remaining graph structure).

3. Similar to the above, any automorphism for the top left graph must map each degree two vertex adjacent to the degree three vertex to another such vertex, and after this the automorphism is determined. As there are only two such maps, the automorphism group has order two.

4. Any map which fixes the degree 5 vertex is an automorphism of the bottom right graph, so that there are $5! = 120$ automorphisms of this graph

5. In an automorphism of the bottom middle graph, the degree 4 vertex must be preserved, and the automorphism is entirely determined by the manner in which it permutes the three degree 1 vertices adjacent to the degree 4 vertex (and each such

permutation induces an automorphism), so that the automorphism group has order $3! = 6$.

6. Finally, the automorphisms of the bottom left graph consist of the maps which send the degree 3 vertices to degree 3 vertices, and preserve the adjacencies to the degree 3 vertices. As there are two degree 3 vertices, and two degree 1 vertices adjacent to each of these (which can be switched in an automorphism), the automorphism group has size $2^3 = 8$.

The following table summarizes these results:

Graph	Size of Automorphism group	Number of isomorphic spanning trees
Top Left	2	$6!/2 = 360$
Top Middle	2	$6!/2 = 360$
Top Right	2	$6!/2 = 360$
Bottom Right	$5!$	$6!/5! = 6$
Bottom Middle	$3!$	$6!/3! = 120$
Bottom Left	2^3	$6!/2^3 = 90$

Note that the total number of spanning trees (computed by summing the right column of the table above) is $1296 = 6^4$, which is consistent with Theorem 2.1 from the text. $\square$