1.) We seek $c, d$ so that

$$(a + bi)(c + di) = (ac - bd) + (ad + bc)i = 1 + 0i.$$ 

Thus we solve the linear system:

$$ac - bd = 1, \quad bc + ad = 0$$

for $c$ and $d$. Manipulation yields $(a^2 + b^2)d = -b, (a^2 + b^2)c = a$ so

$$c + di = \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2}.$$ 

You may recognize this as the complex conjugate of $a + bi$ divided by the square of the norm.

3.) Both $v$ and $-(-v)$ are the additive inverse of $-v$. Since additive inverses are unique, we have that $v = -(-v)$ (proposition 1.3).

4.) If $a \neq 0$ then we can write

$$0 = \frac{1}{a} 0 = \frac{1}{a} (av) = \left( \frac{1}{a} \cdot a \right) v = 1 \cdot v = v.$$ 

It follows that either $a = 0$ or $v = 0$.

5.) In each case, call the set in question $U$.

   a. This is a subspace. We have $0 = \{0,0,0\} \in U$ since $1 \cdot 0 + 2 \cdot 0 + 3 \cdot 0 = 0$. Also if $x = \{x_1, x_2, x_3\}$ and $y = \{y_1, y_2, y_3\}$ are two elements of $U$, and $\alpha$ is a scalar then $x + y = \{x_1 + y_1, x_2 + y_2, x_3 + y_3\}$ satisfies $(x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = (x_1 + 2x_2 + 3x_3) + (y_1 + 2y_2 + 3y_3) = 0 + 0 = 0$ and similarly, $\alpha x = \{\alpha x_1, \alpha x_2, \alpha x_3\}$ satisfies $\alpha x_1 + 2\alpha x_2 + 3\alpha x_3 = \alpha (x_1 + 2x_2 + 3x_3) = \alpha \cdot 0 = 0$. Thus $x + y, \alpha x \in U$ so $U$ has 0 and is closed under addition and scalar multiplication.

   b. This is not a subspace because it is not closed under scalar multiplication. Indeed $\{4,0,0\} \in U$ but $2 \cdot \{4,0,0\} = \{8,0,0\} \not\in U$. In fact, this set fails the other two conditions for a subspace as well.

   c. This is not a subspace because it is not closed under addition. For example, $\{1,0,0\} \in U$ and $\{0,1,1\} \in U$ but $\{1,0,0\} + \{0,1,1\} = \{1,1,1\} \not\in U$. 


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d. This is a subspace. It has 0 since \( \{0,0,0\} \) satisfies 0 = 5 \cdot 0. If \( x = \{x_1,x_2,x_3\} \) and \( y = \{y_1,y_2,y_3\} \) satisfy \( x_1 = 5x_3, y_1 = 5x_3 \) then \( x + y = \{x_1+y_1,x_2+y_2,x_3+y_3\} \) satisfies 

\[
(x_1 + y_1) = (5x_3 + 5y_3) = 5(x_3 + y_3)
\]

as well, so \( U \) is closed under addition. Moreover, if \( \alpha \) is some constant, then \( \alpha x = \{\alpha x_1, \alpha x_2, \alpha x_3\} \) satisfies \( \alpha x_1 = \alpha(5x_3) = 5(\alpha x_3) \) and so \( U \) is closed under scalar multiplication.

8.) Write \( U = \bigcap_c U_c \) where \( U_c \) is a collection of subspaces. Since each \( U_c \) is a subspace, each has 0, and thus 0 is in their intersection. Take \( x, y \in U \) and a scalar \( \alpha \). Then \( x, y \in U_c \) for each \( c \). Since \( U_c \) is closed under addition and scalar multiplication, \( x + y, \alpha x \in U_c \) for each \( c \). Thus \( x + y \) and \( \alpha x \) are contained in \( U \) and \( U \) is closed under addition and scalar multiplication.

9.) We first prove that if \( U \subset W \) are subspaces of \( V \) then \( U \cup W \) is a subspace of \( V \). Indeed, \( U \cup W = W \), and this is a subspace.

To prove the reverse implication, suppose that \( U \) and \( W \) are subspaces so that \( U \not\subset W \) and \( W \not\subset U \). We’ll show that \( U \cup W \) is not a subspace. The fact \( U \not\subset W \) implies there is an element \( u \in U \) not in \( W \). Similarly, \( W \not\subset U \) implies there exists \( w \in W \) but not in \( U \). Since \( u \in U, w \in W \) we have \( u, w \in U \cup W \). But \( u + w \) is not in \( U \) or else additive closure would imply \( (u + w) - u = w \in U \), a contradiction. Similarly, \( u + w \in W \) would force \( u \in W \), another contradiction. Thus \( u + w \) is not in either \( U \) or \( W \) and hence is not in their union. This proves that \( U \cup W \) is not a subspace since it is not closed under addition.

10.) We’ll show \( U + U = U \). Take \( w \in U + U \). Then \( w \) may be written \( w = u_1 + u_2 \) where \( u_1, u_2 \in U \). Since \( U \) is closed under addition, \( w \in U \) and it follows \( U + U \subset U \). To see the reverse inclusion, observe that for all \( u \in U \), \( 0 \in U \) implies \( u + 0 = u \in U + U \). This proves \( U \subset U + U \) so \( U = U + U \).

13.) We provide a counterexample. Let \( V \) be the real vector space \( \mathbb{R}^1 \), \( U_1 \) be the (trivial) subspace \( \{0\} \) and \( U_2 = W = \mathbb{R}^1 \). Any real number can be written as the sum of zero and itself, so \( U_1 + W = \mathbb{R}^1 = U_2 + W \).

14.) Define \( W = \left\{ \sum_{j=0}^{n} a_j z^j : n \geq 6; a_j \in \mathbb{F}; \text{ and } a_2 = a_5 = 0 \right\} \). We show that \( W \) is a subspace of \( \mathcal{P}(\mathbb{F}) \) and \( \mathcal{P}(\mathbb{F}) = U \oplus W \). Take any \( n \) and \( a_j = 0 \) for all \( j \) to see that \( 0 \in W \). Suppose \( p_1(z) = \sum_{j=1}^{n} a_j z^j \), and \( p_2(z) = \sum_{j=1}^{m} b_j z^j \) are two elements of \( W \) and hence have \( a_2 = a_5 = b_2 = b_5 = 0 \). Choose \( M \geq \max(m,n) \). Then setting \( a_i = 0 \) for \( i > n \), \( b_j = 0 \) for
\[ j > m, \text{ we have} \]

\[ p_1(z) + p_2(z) = \sum_{i=0}^{M} a_i z^i + \sum_{j=0}^{M} b_j z^j = \sum_{k=0}^{M} (a_k + b_k) z^k. \]

In this polynomial, \( a_2 + b_2 = a_5 + b_5 = 0 \). Thus \( p_1(z) + p_2(z) \) is a polynomial in \( W \). Again, for \( \alpha \) a constant, \( \alpha p_1(z) = \alpha \sum_{i=1}^{n} a_i z^i = \sum_{i=1}^{n} (\alpha a_i) z^i \). We know \( \alpha a_2 = \alpha a_5 = 0 \) so \( \alpha p_1(z) \) is a member of \( W \) and \( W \) is closed under scalar multiplication.

It remains to show \( \mathcal{P}(F) = U \oplus W \). Take \( p(z) = \sum_{j=0}^{n} a_j z^j \in \mathcal{P}(F) \). Then

\[ p(z) = (a_2 z^2 + a_5 z^5) + \sum_{j \neq 2,5, j \leq n} a_j z^j. \]

This expresses \( p \) as the sum of a polynomial in \( U \) and a polynomial in \( W \) so \( U + W = \mathcal{P}(F) \). Moreover, suppose that \( 0 = p_1(z) + p_2(z) \) with \( p_1 \in U, p_2 \in W \). Then the sum \( p_1 + p_2 \) is a polynomial with all coefficients zero. The only contributions to the coefficients on \( x^2 \) and \( x^5 \) come from \( p_1 \) and so these are zero. The only contribution to other coefficients come from \( p_2 \) and these are also zero. It follows that \( p_1 = p_2 = 0 \) and this proves that the sum \( U + W \) is direct (proposition 1.8).