Finite differences for DE's

This method for solving differential eq'ns (both ordinary and partial) involves (three) steps:

1) Discretize the domain on which the DE holds. We'll use an evenly-spaced grid here for simplicity, but in practice the choice of grid highly impacts the quality of the numerical sol'n.

2) Use appropriate finite difference formulas to approximate derivative terms.

3) Incorporate boundary conditions/initial data.

In any case, this method provides a means of translating a differential eq'n into a system of algebraic eq'ns (if the DE is linear, the resulting system is linear too! A DE is linear if y_1, y_2 are sol'ns and α_1, α_2 are scalars $\Rightarrow \alpha_1 y_1 + \alpha_2 y_2$ is also a sol'n).

The method is perhaps best illustrated by example.

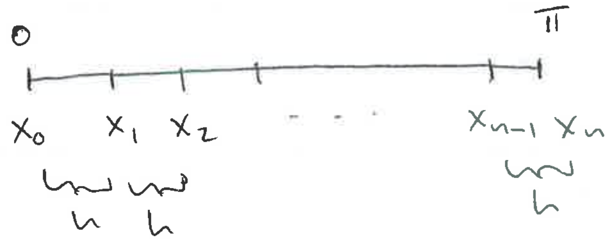
e.g. Given an integer m, solve

BC's which specify the value of the f'n at bdary are Dirichlet

$$v''(x) = -m^2 v(x), \quad 0 < x < \pi, \quad v(0) = v(\pi) = 0$$

1) We use $n+1$ evenly spaced pts. on $[0, \pi]$, and
2) we use the central difference to approximate $v''(x_j)$.

with $x_j = 0 + jh$, $\quad h = \dfrac{\pi - 0}{n}$.



The discretized eq'n becomes

$$\frac{v_{j-1} - 2v_j + v_{j+1}}{h^2} = -m^2 v_j, \quad j = 1, 2, \ldots, n-1$$

only indices corresponding to interior pts!

with $v_j = v(x_j)$ as usual.

Rearranging gives

$$\frac{1}{h^2} v_{j-1} + \left(m^2 - \frac{2}{h^2}\right) v_j + \frac{1}{h^2} v_{j+1} = 0.$$

3) Now we incorporate the boundary data.

Since $x_0 = 0$, $v(0) = 0 \iff v_0 = v(x_0) = v(0) = 0$,

and since $x_n = \pi$, $v(\pi) = 0 \iff v_n = v(\pi) = 0$.

Thus we obtain the (linear) system of $n+1$ eq'ns in $n+1$ variables:

$$\begin{aligned}
v_0 &= 0 \quad \text{left BC} \\
v_0 + (h^2 m^2 - 2) v_1 + v_2 &= 0 \\
v_1 + (h^2 m^2 - 2) v_2 + v_3 &= 0 \\
&\ddots \\
v_{n-2} + (m^2 h^2 - 2) v_{n-1} + v_n &= 0 \\
v_n &= 0 \quad \text{right BC}
\end{aligned}$$

$j = 1, \ldots, n-1$

In fact, this system is tridiagonal! We can write it as

$$(A - \lambda I)\tilde{v} = 0, \quad \text{with}$$

$$A = \begin{bmatrix} -2 & 1 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 1 & -2 \end{bmatrix}, \quad \text{and } \lambda = -(mh)^2, \text{ and } \quad \tilde{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_{n-1} \end{bmatrix}$$

(note A is $(n-1) \times (n-1)$, and we obtain this reduced system by substituting the values $v_0 = 0$ into the eq'n from (2) with $j=1$, and $v_n = 0$ into the eq'n from (2) with $j = n-1$.

Rmk. (cf. HW4 Problem 1 (d)) ~~(this equation system)~~ ~~(implication)~~ If λ is an eigenvalue of A, the system will <u>not</u> have a unique sol'n, since there must be eigenvectors corresponding to λ (and these are non-zero sol'ns to $(A - \lambda I)\tilde{v} = 0$).

Now let's consider an example dealing with a partial differential eq'n.

Example (Laplace eq'n)

The steady-state temperature distribution of a metal plate on $[0,1] \times [0,1]$ is described by the eq'n:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \qquad (x,y) \in (0,1) \times (0,1),$$

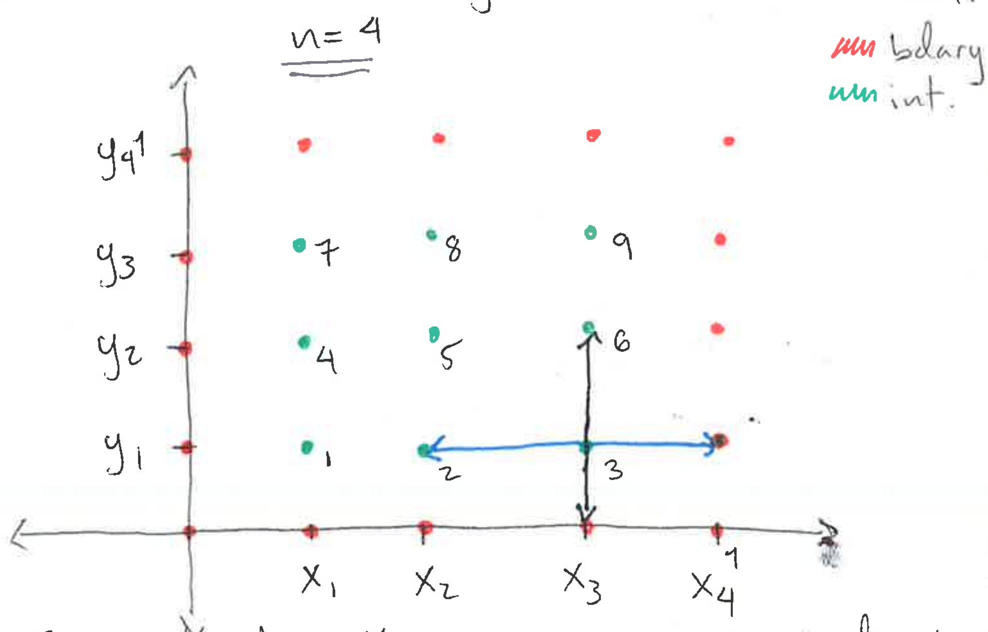
along with the (in this case Dirichlet) BC's:

$$T(0,y) = l(y), \quad T(1,y) = r(y),$$
$$T(x,0) = b(x), \quad T(x,1) = t(x), \qquad \text{where}$$

l, r, b, t are given f'ns which specify the (~~value~~) temperature of the plate at the boundary.

We may ~~(found)~~ follow the same steps as before.

1) Discretize the domain. For simplicity, we use an evenly-spaced square grid, as in the diagram below. We set $x_i = 0 + ih$, $y_j = 0 + jh$, and $h = \frac{1-0}{n}$, after we have chosen the number of points $n+1$ to use along each dimension.



(The numbers next to the green points indicate their index w.r.t. the lexicographical order [cf. step 3 below!])

2) Use central difference to approximate ~~(each)~~ the partial derivatives:

$$\frac{\partial^2 T}{\partial x^2}(x_i, y_j) \approx \frac{T_{i-1,j} - 2T_{i,j} + T_{i+1,j}}{h^2}$$

j is constant, we move along same row on grid

$$\frac{\partial^2 T}{\partial y^2}(x_i, y_j) \approx \frac{T_{i,j-1} - 2T_{i,j} + T_{i,j+1}}{h^2}$$

i is constant, we move along same col'n on grid

Thus for $1 \le i,j \le n-1$, we obtain the equality

$$\frac{T_{i-1,j} - 2T_{ij} + T_{i+1,j}}{h^2} + \frac{T_{i,j-1} - 2T_{ij} + T_{i,j+1}}{h^2} = 0$$

$$\Leftrightarrow T_{i-1,j} + T_{i+1,j} + T_{i,j-1} + T_{i,j+1} - 4T_{ij} = 0 \qquad (*)$$

3) Incorporate BC's.

~~(TODO)~~ Since we are dealing with Dirichlet BC's, incorporating the boundary data to our system is a matter of substituting known values of T_{kl} into $(*)$ when $k = 0$ or n, and when $l = 0, n$. This substitution will modify the "load vector" (RHS vector in linear system $Ax = b$).

The key step in constructing the matrix corresponding to this linear system is traversing the grid points in the "right order." Perhaps the most natural order is the lexicographical order, shown in the diagram above:

the grid point ~~a~~ (x_i, y_j) gets index $k = i + (j-1)(n-1)$.

Then, for each $i = 1, \dots, n-1$,
for each $j = 1, \dots, n-1$,
the kth row in our system is given by eq'n $(*)$,
with $k = i + (j-1)(n-1)$.

To build the matrix it remains to find for each such k the locations l where $a_{kl} \neq 0$ (note ~~from the st~~) there are at most five of these).
For this task, the stencil diagram is quite helpful.

Rmk. The resulting matrix here has a nice block structure: cf. HW7 sol'n's.