Interpolation Examples

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Lagrange Polynomials

1. *(Lagrange Polynomials, Bradie 5.2.)* Consider the following seven ($n = 6$) interpolating points:

$$x_0 = 0.0,\ x_1 = 1.6,\ x_2 = 3.8,\ x_3 = 4.5,\ x_4 = 6.3,\ x_5 = 9.2,\ x_6 = 10.0.$$

Based on these points, two of the Lagrange basis polynomials

$$L_{6,i}(P) = \prod_{j=0, j\neq i}^{6} \frac{(P - x_j)}{(x_i - x_j)}$$

are

$$\begin{aligned} L_{6,1}(x) =& \frac{x(x-3.8)(x-4.5)(x-6.3)(x-9.2)(x-10.0)}{(1.6)(1.6-3.8)(1.6-4.5)(1.6-6.3)(1.6-9.2)(1.6-10.0)}, \\ L_{6,3}(x) =& \frac{x(x-1.6)(x-3.8)(x-6.3)(x-9.2)(x-10.0)}{4.5(4.5-1.6)(4.5-3.8)(4.5-6.3)(4.5-9.2)(4.5-10.0)}. \end{aligned}$$

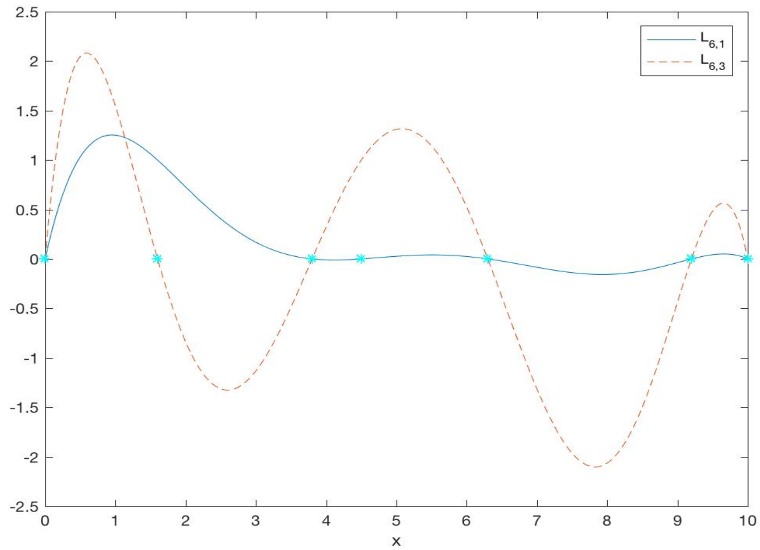
These two polynomials are plotted in Figure 1. The positions of the interpolating points are denoted by $*$.

Note the large amplitude oscillations present in Figure 1. This behavior is characteristic of high-degree polynomials and tends to get worse as the degree of the polynomial is increased. This suggests that in many cases merely increasing the number of interpolating data points may worsen the approximation error at certain points.

Moreover, the plot tells a cautionary tale. Whenever high-degree polynomials are used for interpolation, some sort of consistency check is required. This could involve plotting the data points on the same axes as the interpolating polynomial, for visual verification. Alternatively, one could split the data set, use a portion for building the interpolating polynomial ("training" the model) and then use the reserved data points to measure interpolation accuracy ("testing" the model).

2. *(Interpolation from thermodynamic tables, Bradie 5.1 and 5.3.)* A thermodynamics student needs to determine whether Freon-12 under a pressure of $P = 400$ kPa and with specific volume (volume per unit mass) of $v = 0.042$ m^3/kg is in a saturated or a superheated state. The answer to this question depends upon how the specific volume of $v = 0.042$ m^3/kg compares with the specific volume of saturated Freon-12 vapor, v_g, at a pressure of 400kPa.

Figure 1: Two Lagrange polynomials defined by the given sequence of interpolating points.



If the given vapor pressure is below v_g then the Freon-12 is in a saturated state; otherwise it is in a superheated state.

The available thermodynamic tables (Table A.2.3 of *Fundamentals of Classical Thermodynamics* by Van Wylen and Sonntag) provide the following values for the specific volume of saturated Freon-12 vapor as a function of pressure:

Pressure (kPa)	308.6	362.6	423.3	491.4
v_g (m^3/kg)	0.055389	0.047485	0.040914	0.035413

Using the four ($n = 3$) data points, we can form a third-degree Lagrange interpolating functions, as a linear combination of the appropriate basis $L_{3,i}(P) = \prod_{j=0, j\neq i}^{3} \frac{(P - x_j)}{(x_i - x_j)}$, for

$i = 0, 1, 2, 3$ and x_j taken from the table. By definition, and using the above data, we have:

$$
\begin{aligned}
L_{3,0}(P) =& \frac{(P - x_1)(P - x_2)(P - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)} \\
=& \frac{(P - 362.6)(P - 423.3)(P - 491.4)}{(308.6 - 362.6)(308.6 - 423.3)(308.6 - 491.4)} \\
=& \frac{(P - 362.6)(P - 423.3)(P - 491.4)}{(-54)(-114.7)(-182.8)} \\
L_{3,1}(P) =& \frac{(P - x_0)(P - x_2)(P - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)} \\
=& \frac{(P - 308.6)(P - 423.3)(P - 491.4)}{(362.6 - 308.6)(362.6 - 423.3)(362.6 - 491.4)} \\
=& \frac{(P - 308.6)(P - 423.3)(P - 491.4)}{(54)(-60.7)(-128.8)}
\end{aligned}
$$

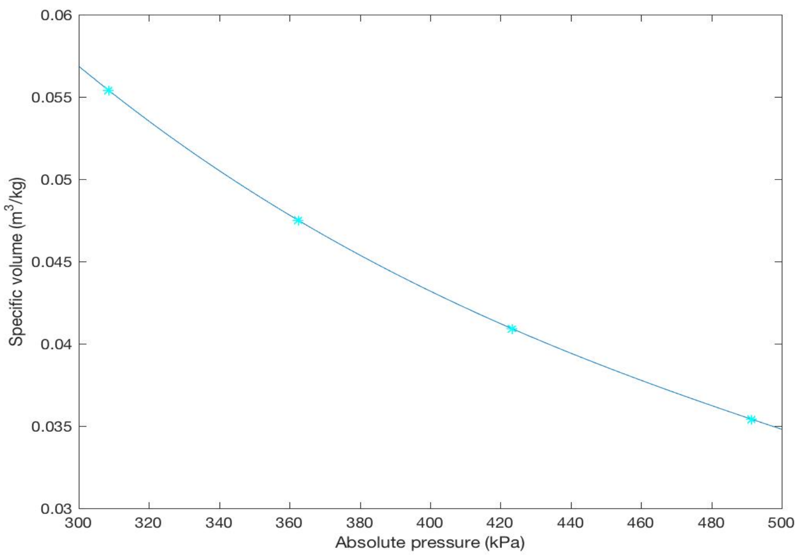
$$
\begin{aligned}
L_{3,2}(P) =& \frac{(P - x_0)(P - x_1)(P - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)} \\
=& \frac{(P - 308.6)(P - 362.6)(P - 491.4)}{(423.3 - 308.6)(423.3 - 363.6)(423.3 - 491.4)} \\
=& \frac{(P - 308.6)(P - 362.6)(P - 491.4)}{(114.7)(60.7)(-68.1)} \\
L_{3,3}(P) =& \frac{(P - x_0)(P - x_1)(P - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)} \\
=& \frac{(P - 308.6)(P - 362.6)(P - 423.3)}{(491.4 - 308.4)(491.4 - 362.6)(491.4 - 423.3)} \\
=& \frac{(P - 308.6)(P - 362.6)(P - 423.3)}{(182.8)(128.8)(68.1)}.
\end{aligned}
$$

Then, the interpolation polynomial is given by:

$$
\begin{aligned}
v_g(P) =& \sum_{i=0}^{n} y_i L_{3,i}(P) \\
=& \frac{(P - 362.6)(P - 423.3)(P - 491.4)}{(-54)(-114.7)(-182.8)} \cdot 0.055389 + \frac{(P - 308.6)(P - 423.3)(P - 491.4)}{(54)(-60.7)(-128.8)} \cdot 0.047485 \\
+& \frac{(P - 308.6)(P - 362.6)(P - 491.4)}{(114.7)(60.7)(-68.1)} \cdot 0.040914 + \frac{(P - 308.6)(P - 362.6)(P - 423.3)}{(182.8)(128.8)(68.1)} \cdot 0.035413.
\end{aligned}
$$

The interpolating polynomial, plotted in Figure 2, clearly provides a plausible representation of the data. Using this polynomial, we find the specific volume of saturated Freon-12 vapor at a pressure of 400kPa by evaluating $v_g(400) = 0.043199 \text{ m}^3/\text{kg}$, thereby solving the problem at hand.

Figure 2: Third-degree interpolating polynomial for specific volume as a function of absolute pressure. Data points used to construct the interpolation are denoted by $*$.



3. *(Emittance of Tungsten as Function of Temperature.)* The table below gives experimental values for the emittance of tungsten as a function of temperature.

Temperature (10^2 K)	3	4	5	6	7	8	9	10	11
Emittance	0.024	0.035	0.046	0.058	0.067	0.083	0.097	0.111	0.125

The eighth-degree Lagrange interpolant is plotted in Figure 3. Note the oscillating behavior of the polynomial, in the ranges $300 - 500$K and $900 - 1100$K. As mentioned in a previous example, this behavior is typical of high-degree interpolations and does not seem to be very consistent with the underlying given data. The use of a spline interpolant (piecewise low-degree polynomial) would be advisable for this problem.

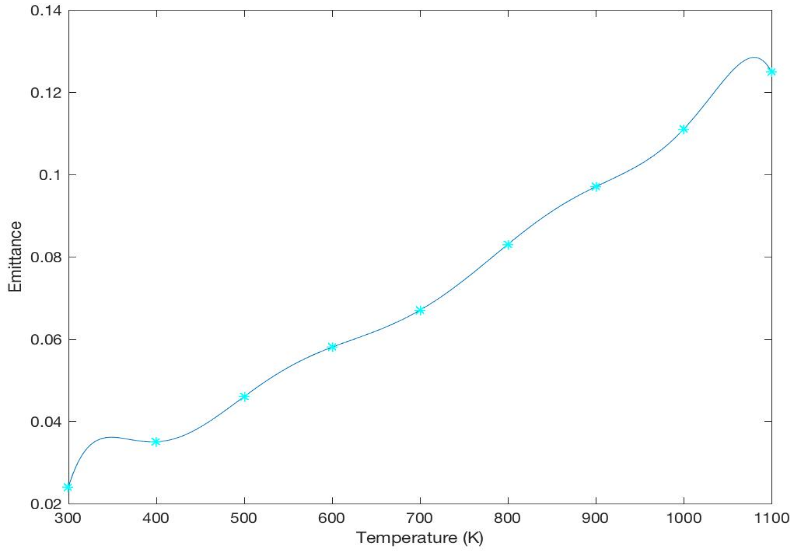
4. *(Runge's function)* Consider the problem of interpolating Runge's function

$$f(x) = \frac{1}{1 + x^2}$$

in the interval $[-5, 5]$. We choose 11 equally spaced points in the interval and form the Lagrange form of the interpolating polynomial using MATLAB. Refer to the code below for a very naive $O(n^3)$ implementation. For a more efficient implementation, please refer to the barycentric interpolation method discussed in lecture. Our results are plotted in Figure 4.

```
n = 50;
N = 1001;
```

Figure 3: Eighth-degree interpolating polynomial for emittance of tungsten as a function temperature. Each data point used to construct the interpolating function is denoted by $*$.



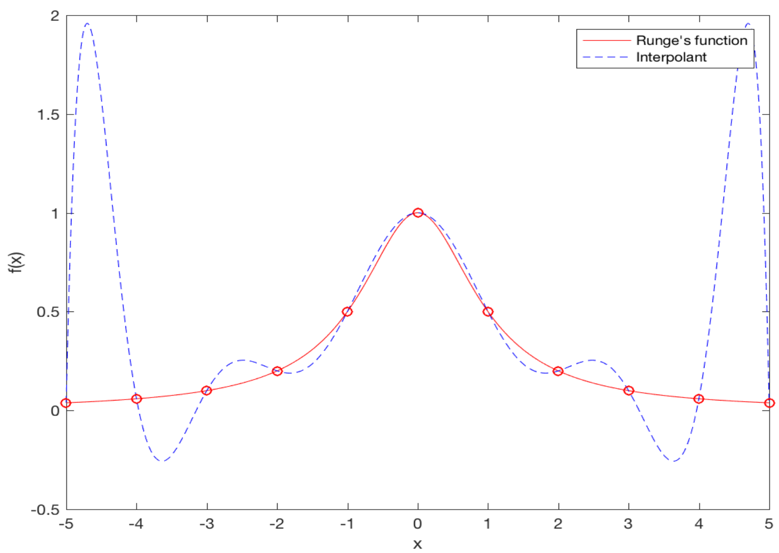

```

%Plot true function:
x = linspace(-5,5,N);
y_true = 1./(1+x.^2);
plot(x, y_true, '-r')
hold on

%a) Lagrange polynomial with equally spaced points:
xi = linspace(-5,5,n);
fi = 1./(1 + xi.^2);
y_equal = zeros(1,N);
for j=1:N
    for i=1:n
        L = 1;
        for k=1:n
            if k~=i
                L = L*(x(j)-xi(k))/(xi(i)-xi(k));
            end
        end
        y_equal(j) = y_equal(j)+fi(i)*L;
    end
end
plot(x,y_equal,'--b')
```

```
xlabel('x')
ylabel('f(x)')
```

Figure 4: Tenth-degree polynomial interpolating Runge's function. Abscissas and corresponding function values are marked by red circles.



Note the wild oscillations of the interpolating polynomial at the edges of the interval. This is characteristic of high-degree polynomial interpolants, and precisely the issue which piecewise (spline) interpolants aim to resolve!