

### Finite difference method for ODE – Solution

Consider the following boundary value problem

$$(x-1)y'' - xy' + y = -(1-x)^2; \quad x \in [0, 2]$$
$$y(0) = 0; \quad y(2) + \frac{1}{2}y'(2) = -\frac{1}{4}.$$

Using  $N = 41$  node points,

1. Derive the tri-diagonal system to solve the ODE using central difference for both 1st- and 2nd-derivatives. Identify elements of vectors  $a$ ,  $b$ ,  $c$  and  $f$ .
2. Plot your numerical solution against analytical solutions. You can obtain the analytical solution from Wolfram-Alpha, and evaluate the arbitrary constants.

#### Solution

Applying central difference formulas and general node  $j$ :

$$(x_j - 1) \frac{y_{j-1} - 2y_j + y_{j+1}}{h^2} - x_j \frac{y_{j+1} - y_{j-1}}{2h} + y_j = -(1 - x_j)^2$$

or, multiplying by  $h^2$  and rearranging:

$$[(1 + 0.5h)x_j - 1] y_{j-1} + [h^2 - 2(x_j - 1)] y_j + [(1 - 0.5h)x_j - 1] y_{j+1} = -h^2(1 - x_j)^2$$

Thus,

$$a_j = (1 + 0.5h)x_j - 1;$$
$$b_j = h^2 - 2(x_j - 1);$$
$$c_j = (1 - 0.5h)x_j - 1;$$
$$f_j = -h^2(1 - x_j)^2$$

At  $j = 2$ ,  $y_1 = 0$ ; thus,

$$[h^2 - 2(x_2 - 1)] y_2 + [1 - 0.5h)x_2 - 1] y_3 = -h^2(1 - x_2)^2$$

At  $j = N$ , we have  $y_N + \frac{y_{N+1} - y_{N-1}}{4h} = -\frac{1}{4}$ ; and

$$(x_N - 1)y_{N-1} + [0.5h^2 - (x_N - 1) - 2h((1 - 0.5h)x_N - 1)]y_N = -0.5h^2(1 - x_N)^2 + 0.5h[(1 - 0.5h)x_N - 1]$$

Modified coefficients for the last equation:

$$a_N = x_N - 1;$$

$$b_N = 0.5h^2 - (x_N - 1) - 2h[(1 - 0.5h)x_N - 1];$$

$$f_N = -0.5h^2(1 - x_N)^2 + 0.5h[(1 - 0.5h)x_N - 1]$$

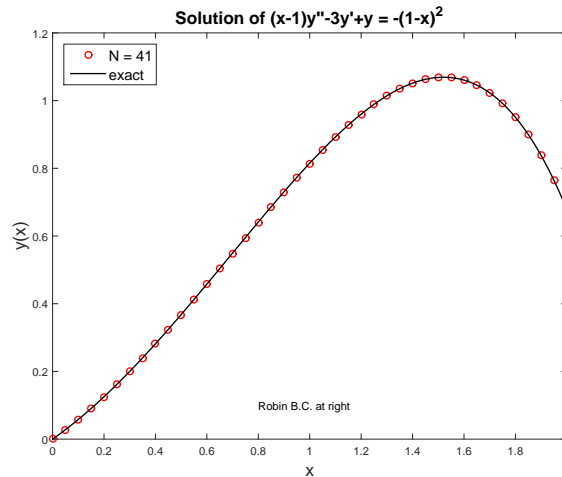
The exact solution is

$$y = C_1x + C_2e^x + x^2 + 1 \quad \text{and} \quad y' = C_1 + C_2e^x + 2x$$

Applying boundary conditions gives

$$C_1 = \frac{3}{5}e^2 - \frac{29}{10} \approx 1.5334; \quad C_2 = -1$$

### Results and Script



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1 function ode_direct_finite_diff_robin
2     % solve the ODE
3     % (x-1)y''-xy'+y = -(1-x)^2

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4      %
5  close all
6  %
7      N = 41;                % number of discrete points
8      yL = 0.0;             % left boundary condition
9  %
10     xL = 0;                % left boundary
11     xR = 2;                % right boundary
12     h = (xR-xL)/(N-1);    % grid spacing
13     x = linspace(xL,xR,N); % x(j) for plots
14 %
15     alpha = 1; beta = 0.5; gamma = -.25; % Robin
16     A = [xL exp(xL) ; alpha*xR+beta (alpha+beta)*exp(xR)];
17     d = [yL-xL^2-1 ; gamma-alpha*(xR^2+1)-2*beta*xR];
18     C = A\d;
19 %
20 % Calculate tridiag matrix elements for interior points j
    = 2,3,...,N-1
21 %
22     c(1:N-1) = (1-0.5*h)*x(2:N)-1;
23 %
24     a(1:N-1) = (1+0.5*h)*x(2:N)-1;
25 %     a(N-1) = beta*(a(N-1)+c(N-1));
26     a(N-1) = x(N)-1;
27 %
28     b(1:N-1) = h^2-2*(x(2:N)-1);
29 %     b(N-1) = beta*b(N-1)-2*alpha*h*c(N-1);
30     b(N-1) = 0.5*h^2-(x(N)-1)-2*h*((1-0.5*h)*x(N)-1);
31 %
32 % Calculate right hand side vector (source terms)
33 %
34     f(1) = -h^2*(1-x(2))^2-((1+0.5*h)*x(2)-1)*yL;
35     f(2:N-1) = -h^2*(1-x(3:N)).^2;
36 %     f(N-1) = beta*f(N-1)-2*gamma*h*c(N-1);
37     f(N-1) = -0.5*h^2*(1-x(N))^2+0.5*h*((1-0.5*h)*x(N)-1);
38 %
39 % Call 'tridiag' to solve for y
40 y = tridiag(N-1,a,b,c,f);

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41 % y vector has N-1 interior points, add yL at left to
    complete
42 y = [yL y];
43 %
44 % calculate exact solution for comparison
45 x1 = linspace(xL,xR,1025);
46 yexact = C(1)*x1+C(2)*exp(x1)+x1.^2+1;
47 plot(x(1:N),y(1:N),'or',x1,yexact,'k','linewidth',1.15)
48 %
49 h_leg = legend(['N = ',num2str(N)],'exact','Location','
    NorthWest');
50 set(h_leg,'fontsize',13)
51 xlabel('x','fontsize',15)
52 ylabel('y(x)','fontsize',15)
53 title('Solution of (x-1)y''''-3y'''+y = -(1-x)^2','fontsize',
    ,15)
54 %
55 text(.8,0.1,'Robin B.C. at right')
56 %print(figure(1),'-dpdf','finite_diff_robin.pdf')
57 end
58 %
59 function x = tridiag(n,a,b,c,f)
60 for j = 2:n
61     b(j) = b(j)-a(j)/b(j-1)*c(j-1);
62     f(j) = f(j)-a(j)/b(j-1)*f(j-1);
63 end
64 x(n) = f(n)/b(n);
65 for j=n-1:-1:1
66     x(j) = (f(j)-c(j)*x(j+1))/b(j);
67 end
68 end

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