Problem set 3
due Mon, 1/30 in class

(1) (10 points) Recall that \( \mathcal{L}_0 = \{ A \subset \mathbb{R}^n : \lambda_*(A) = \lambda^*(A) < \infty \} \) and \( \mathcal{L} = \{ A \subset \mathbb{R}^n : A \cap M \in \mathcal{L}_0 \text{ for all } M \in \mathcal{L}_0 \} \).

Let \( M_1 \subset M_2 \subset \ldots \subset \mathbb{R}^n \) be an increasing sequence with \( M_i \in \mathcal{L}_0 \) and \( \lim \sup_{i \to \infty} M_i = \bigcup_{i=1}^{\infty} M_i = \mathbb{R}^n \). Show that \( A \in \mathcal{L} \) if and only if \( A \cap M_i \in \mathcal{L}_0 \) for all \( i \geq 1 \). (In particular this implies that \( A \in \mathcal{L} \) if and only if \( A \cap B(0, i) \in \mathcal{L}_0 \) for all \( i \geq 1 \).)

(2) (10 points) Let \( A_1, A_2, \ldots \in \mathcal{L} \) and let \( A = \bigcup_{i=1}^{\infty} A_i \) be their union. Assume that \( \lambda(A) < \infty \). Show that

\[
\lambda(A) = \sum_{i=1}^{\infty} \lambda(A_i)
\]

if and only if \( \lambda(A_i \cap A_j) = 0 \) for all distinct \( i, j \geq 1 \).

(3) (10 points) Let \( A \subset \mathbb{R}^n \). Show that \( A \in \mathcal{L} \) if and only if one of the following conditions is satisfied:

(a) There is an increasing sequence \( K_1 \subset K_2 \subset \ldots \) of compact sets \( K_i \in \mathcal{K} \) with \( K_i \subset A \) such that \( \lambda^*(A \setminus \bigcup_{i=1}^{\infty} K_i) = 0 \).

(b) There is a decreasing sequence \( G_1 \supset G_2 \supset \ldots \) of open sets \( G_i \in \mathcal{O} \) with \( A \subset G_i \) such that \( \lambda^*(\bigcap_{i=1}^{\infty} G_i \setminus A) = 0 \).

(4) (30 points) Recall that for any invertible matrix \( T \in \mathbb{R}^{n \times n} \) and vector \( z \in \mathbb{R}^n \), the affine map \( \Phi_{T,z} : \mathbb{R}^n \to \mathbb{R}^n \) is defined by \( \Phi_{T,z}(x) = Tx + z \).

(a) Assume in (a)-(d) that \( T = \text{diag}(\lambda_1, \ldots, \lambda_n) \), where the \( \lambda_i \) do not need to be positive. Show that for any \( P \in \mathcal{F} \) and \( \varepsilon > 0 \) there is a \( P' \in \mathcal{F} \) with

\[
||\det T|\lambda_1(P) - \lambda_1(P')|| < \varepsilon
\]

such that \( P' \subset \Phi_{T,0}(P) \).

(b) Conclude from (a) that for any open set \( A \in \mathcal{O} \), we have \( \lambda_2(\Phi_{T,0}(A)) = |\det T|\lambda_2(A) \).

(c) Conclude from (b) that for any compact set \( A \in \mathcal{K} \), we have \( \lambda_3(\Phi_{T,0}(A)) = |\det T|\lambda_3(A) \).

(d) Conclude from (b) and (c) that for any \( A \subset \mathbb{R}^n \) we have \( \lambda^*(\Phi_{T,0}(A)) = |\det T|\lambda^*(A) \) and \( \lambda_*(\Phi_{T,0}(A)) = |\det T|\lambda_*(A) \). Conclude from this that for
any $A \subset \mathbb{R}^n$ we have $A \in \mathcal{L}_0$ if and only if $\Phi_{T,0}(A) \in \mathcal{L}_0$ and $A \in \mathcal{L}$ if and only if $\Phi_{T,0}(A) \in \mathcal{L}$. Moreover, show that in those cases $\lambda(\Phi_{T,0}(A)) = |\det T|\lambda(A)$.

(e) Now assume in (d) and (e) that $T$ is an elementary matrix, i.e. it is the sum of the identity matrix $E_n$ and a matrix which has only one nonzero entry $c$ off the diagonal. Let $Q = [a_1, b_1] \times \ldots \times [a_n, b_n] \subset \mathbb{R}^n$ be a special rectangle and set $Q' = \Phi_{T,0}(Q)$. Find decompositions $Q = Q'_1 \cup \ldots \cup Q'_k$ and $Q' = Q'_1 \cup \ldots \cup Q'_k$ as well as $z_1, \ldots, z_k \in \mathbb{R}^n$ such that $Q_i = \Phi_{E_n, z_i}(Q_i)$ for $i = 1, \ldots, k$. Moreover, make sure that $Q_i, Q'_i \in \mathcal{L}_0$. Conclude from this that $\lambda(Q') = \lambda(Q) = \lambda_1(Q)$.

(Hint: If you are looking for an idea, try to do everything in dimension $n = 2$ and draw a picture. You might have to distinguish the cases $c < 0$ and $c > 0$. Make sure that you indicated which boundary components of $Q_i$ and $Q'_i$ are included or excluded.)

(f) Use (e) to show that for any $P \in \mathcal{F}$, we have $\lambda(\Phi_{T,0}(P)) = \lambda_1(P)$.

(g) Show that any invertible matrix $T \in \mathbb{R}^{n \times n}$ is the product of an invertible diagonal matrix $D \in \mathbb{R}^{n \times n}$ and elementary matrices $T_1, \ldots, T_k \in \mathbb{R}^{n \times n}$:

$$T = DT_1 \cdots T_k.$$  

(Hint: Use the algorithm for the reduced row echelon form.)

5 (20 points) Recall that a function $f : D \subset \mathbb{R}^m \to \mathbb{R}^n$ is called Lipschitz if there is a constant $L$ such that $|f(x) - f(y)| \leq L|x - y|$ for all $x, y \in D$. Moreover, $f$ is called locally Lipschitz if for all compact subsets $K \subset D$, the restriction $f|_K$ is Lipschitz.

(a) Assume $n \geq 2$ and consider a Lipschitz function $f : [a, b] \to \mathbb{R}^n$. Let $A = f([a, b])$ be its image. Show that $A \in \mathcal{L}$ and $\lambda(A) = 0$. 

(Hint: Cover the image by small balls.)

(b) More generally, assume that $m < n$ and consider a Lipschitz function $f : [a_1, b_1] \times [a_m, b_m] \to \mathbb{R}^n$ and set $A = f([a_1, b_1] \times \ldots \times [a_m, b_m])$. Show again that $A \in \mathcal{L}$ and $\lambda(A) = 0$.

(c) Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a locally Lipschitz function, $m < n$ and let $A = f(\mathbb{R}^m) \subset \mathbb{R}^n$ its image. Show again that $A \in \mathcal{L}$ and $\lambda(A) = 0$.

(Hint: Use (b) and countable additivity.)

Maximum total points: 80