MATH 172: LEBESGUE INTEGRATION AND FOURIER ANALYSIS
(WINTER 2012)

Problem set 7

due Mon, 3/5 in class (note the change!)

(1) (20 points) The following principle is a very useful tool to prove a large variety of inequalities:

The minimum of a sum of functions is greater or equal to the sum of their minima.

(a) Use the functions below on appropriate domains to establish the inequality between the harmonic, geometric, arithmetic and quadratic mean: For any positive real numbers $a_1, \ldots, a_n$ we have

$$\frac{1}{a_1} + \ldots + \frac{1}{a_n} \leq \sqrt[n]{a_1 \cdots a_n} \leq \frac{a_1 + \ldots + a_n}{n} \leq \sqrt[2]{\frac{a_1^2 + \ldots + a_n^2}{n}}.$$ 

When do we have equality?

Functions: $f_i(x) = (x + a_i)^2$, $f_i(x) = a_i x - \log x$, $f_i(x) = \frac{x}{a_i} - \log x$.

(b) Use the same principle for weighted sums, to show the following weighted inequality: For any positive real numbers $a_1, \ldots, a_n$ and $m_1, \ldots, m_n$, we have

$$\frac{a_1 + \ldots + a_n}{m_1 a_1 + \ldots + m_n a_n} \leq \left(\frac{a_1^{m_1} \cdots a_n^{m_n}}{m_1 \cdots + m_n}\right)^{1/m_1 + \ldots + m_n} \leq \frac{m_1 a_1 + \ldots + m_n a_n}{m_1 + \ldots + m_n} \leq \sqrt[2]{\frac{m_1 a_1^2 + \ldots + m_n a_n^2}{m_1 + \ldots + m_n}}.$$ 

When do we have equality?

(c) Prove HÖLDER’s inequality using the functions below: For any nonnegative real numbers $a_1, \ldots, a_n, b_1, \ldots, b_n$ and any $p, q \in (1, \infty)$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$a_1 b_1 + \ldots + a_n b_n \leq (a_1^p + \ldots + a_n^p)^{1/p} (b_1^q + \ldots + b_n^q)^{1/q}.$$ 

In the case $p = q = 2$, this inequality is called CAUCHY-SCHWARZ inequality. When does equality occur?
In this problem we generalize the results from (1) to integrals. First, find an analogous principle in this setting. Let \( M \) and \( \lambda \)

(d) Use Hölder’s inequality to establish the following inequality between general means: For \( 0 < p < q < \infty, a_1, \ldots, a_n \geq 0 \) and \( m_1, \ldots, m_n > 0 \) we have

\[
\left( \frac{m_1a_1^p + \ldots + m_na_n^p}{m_1 + \ldots + m_n} \right)^{1/p} \leq \left( \frac{m_1a_1^q + \ldots + m_na_n^q}{m_1 + \ldots + m_n} \right)^{1/q}.
\]

When do we have equality?

(e) Show that for \( p \to \infty \) the left hand side in (d) converges to the maximum and for \( p \to 0 \) to the weighted geometric mean, i.e.

\[
\lim_{p \to \infty} \left( \frac{m_1a_1^p + \ldots + m_na_n^p}{m_1 + \ldots + m_n} \right)^{1/p} = \max\{a_1, \ldots, a_n\}
\]

\[
\lim_{p \to 0} \left( \frac{m_1a_1^p + \ldots + m_na_n^p}{m_1 + \ldots + m_n} \right)^{1/p} = (a_1^{m_1} \ldots a_n^{m_n})^{1/m_1 + \ldots + m_n}
\]

(2) (20 points) In this problem we generalize the results from (1) to integrals. First, find an analogous principle in this setting. Let \( \mathcal{M} \) be a \( \sigma \)-algebra on \( X \) and \( \lambda \) a measure on \( \mathcal{M} \).

(a) Let \( f : X \to \mathbb{R} \) be an \( \mathcal{M} \)-measurable, integrable function. Show that

\[
\left| \int f \, d\lambda \right| \leq \int |f| \, d\lambda.
\]

(b) Show Hölder’s inequality using the same functions as in (1)(c): Assume that \( p, q \in (1, \infty) \) such that \( \frac{1}{p} + \frac{1}{q} = 1 \). Let \( f, g : X \to \mathbb{R} \) be two \( \mathcal{M} \)-measurable functions such that \( \|f\|_p = (\int |f|^p d\lambda)^{1/p}, \|g\|_q = (\int |g|^q d\lambda)^{1/q} < \infty \). Then \( fg : X \to \mathbb{R} \) is integrable and

\[
\left| \int fg \, d\lambda \right| \leq \left( \int |f|^p d\lambda \right)^{1/p} \left( \int |g|^q d\lambda \right)^{1/q} = \|f\|_p \|g\|_q.
\]

(c) Assume that \( \lambda(X) < \infty \) and \( 0 < p < q < \infty \). Let \( f : X \to \mathbb{R} \) be an \( \mathcal{M} \)-measurable function and assume that \( |f|^p \) is integrable. Use Hölder’s inequality to show that then also \( |f|^q \) is integrable and

\[
(\lambda(X))^{-1/p} \|f\|_p = \left( \frac{1}{\lambda(X)} \int |f|^p d\lambda \right)^{1/p} \leq \left( \frac{1}{\lambda(X)} \int |f|^q d\lambda \right)^{1/q} = (\lambda(X))^{-1/q} \|f\|_q.
\]

When do we have equality?

(d) Assume again that \( \lambda(X) < \infty \) and let \( f : X \to \mathbb{R} \) be an \( \mathcal{M} \)-measurable function. Recall that

\[
\|f\|_\infty = \inf \{ M \in \mathbb{R} : |f| \leq M \ \text{a.e.} \}.
\]

Show that

\[
\lim_{p \to \infty} \left( \frac{1}{\lambda(X)} \int |f|^p d\lambda \right)^{1/p} = \lim_{p \to \infty} (\lambda(X))^{-1/p} \|f\|_p = \|f\|_\infty.
\]
(e) Show that for all $0 < p < q < \infty$ there are functions $f$ on $\mathbb{R}$ such that
\[ \int |f|^q d\lambda < \infty, \text{ but } \int |f|^p d\lambda = \infty. \] (Hence, the condition $\lambda(X) < \infty$ in (c) is in fact necessary.)

(f) Show that the inequalities from (1)(c) and (1)(d) follow immediately from (2)(b) and (2)(c) for the right choice of $X, \mathcal{M}$ and $\lambda$.

(3) (0 points) You don’t need to solve this problem. I forgot that we did something similar on the last problem set. Sorry for that!

Show that for any $p \in [1, \infty)$ there is a sequence of functions $f_1, f_2, \ldots \in L^p(\mathbb{R}^n)$ such that $\|f_i\|_p = 1$ and $f_i = 0$ almost everywhere on $\mathbb{R}^n \setminus B(0,1)$ for all $i$, but $f_i \to 0$ almost everywhere as $i \to \infty$.

(4) (10 points) Let $(V, | \cdot |)$ be a normed space and $y_1, y_2, \ldots \in V$ a Cauchy sequence with respect to $| \cdot |$. Show that there is a subsequence $z_1 = y_{i_1}, z_2 = y_{i_2}, \ldots$ such that $|z_{k+1} - z_k| < 2^{-k}$.

(5) (20 points) Let $\mathcal{M}$ be a $\sigma$-algebra on $X$ and $\lambda$ a measure on $\mathcal{M}$.

(a) Let $p \in [1, \infty)$ and $f \in L^p(X, \mathcal{M}, \lambda)$. Show that for any $\varepsilon > 0$ there is a set $E \in \mathcal{M}$ with $\lambda(E) \leq \varepsilon$ such that
\[ |f(x)| < \varepsilon^{-1/p} \|f\|_p \quad \text{for all } x \in X \setminus E. \]

(b) Let $f \in L^\infty(X, \mathcal{M}, \lambda)$. Show that $|f| \leq \|f\|_\infty$ almost everywhere.

(c) Prove Egorov’s Theorem: Assume that $\lambda(X) < \infty$. Let $f_1, f_2, \ldots : X \to \mathbb{R}$ be a sequence of $\mathcal{M}$-measurable functions such that $\lim_{i \to \infty} f_i = f$ pointwise. Then for any $\varepsilon > 0$ there is a set $X_\varepsilon \in \mathcal{M}$ with $\lambda(X \setminus X_\varepsilon) < \varepsilon$ such that the sequence $f_1, f_2, \ldots$ converges uniformly on $X_\varepsilon$.

(Hint: Consider the sets $X_{i,k} = \{x \in X : |f(x) - f_i(x)| < \frac{1}{k}\}$.)

(6) (20 points) Consider Euclidean space $\mathbb{R}^n$ with the Lebesgue $\sigma$-algebra $\mathcal{L}$ and the Lebesgue measure $\lambda$. In this problem we show that integrable simple functions can be approximated by smooth functions of compact support.

(a) Show that for any $A \in \mathcal{L}$ with $\lambda(A) < \infty$ there is a special polygon $P \in \mathcal{F}$ such that $\lambda(A \setminus P) + \lambda(P \setminus A) < \varepsilon$.

(b) Show that for any $A \in \mathcal{L}$ with $\lambda(A) < \infty$ and any $\varepsilon > 0$, there is a smooth function with compact support $f \in C_0^\infty(\mathbb{R}^n)$ with $0 \leq f(x) \leq 1$ for all $x \in \mathbb{R}^n$ and a set $E \in \mathcal{L}$ such that $\lambda(E) < \varepsilon$ and $f(x) = \chi_A(x)$ for all $x \in \mathbb{R}^n \setminus E$.

(Hint: First approximate $\chi_Q$ by a smooth function where $Q$ is a special rectangle.)

(c) Let $s : \mathbb{R}^n \to \mathbb{R}$ be a simple function and $p \in [1, \infty)$ such that $\int |s|^p d\lambda < \infty$ and $\varepsilon > 0$. Show that there is a smooth function with compact support $f \in C_0^\infty(\mathbb{R}^n)$ such that $\|f - s\|_p < \varepsilon$.

Moreover, we can choose $f$ such that there is a set $E \in \mathcal{L}$ such that $\lambda(E) < \varepsilon$ and $f(x) = s(x)$ for all $x \in \mathbb{R}^n \setminus E$.

(d) Show that for any function $f \in L^p(\mathbb{R}^n)$ (where $p \in [1, \infty)$) there is a nullset $N \in \mathcal{L}$ and a sequence of smooth functions with compact support $f_1, f_2, \ldots \in C_0^\infty(\mathbb{R}^n)$ such that $\lim_{i \to \infty} f_i(x) = f(x)$ for all $x \in \mathbb{R}^n \setminus N$ (i.e. we have pointwise convergence almost everywhere).

Maximum total points: 90