MATH 172: LEBESGUE INTEGRATION AND FOURIER ANALYSIS
(WINTER 2012)

Problem set 8

due Mon, 3/12 in class

(1) (5 points) Let $f_1, f_2 : \mathbb{R}^n \rightarrow \mathbb{R}$ be continuous functions and assume that $f_1 = f_2$ almost everywhere. Show that then $f_1 = f_2$ everywhere.

(2) (5 points) Consider the function $f : (0, 1) \times (0, 1) \rightarrow \mathbb{R}$, $f(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$.

Show that for all $x \in (0, 1)$, the function $y \mapsto f(x, y)$ is $\mathcal{B}$-measurable and integrable and for all $y \in (0, 1)$, the function $x \mapsto f(x, y)$ is $\mathcal{B}$-measurable and integrable. However

$$\int_0^1 \left( \int_0^1 f(x, y) dx \right) dy \neq \int_0^1 \left( \int_0^1 f(x, y) dy \right) dx.$$ 

Why does this not contradict Fubini’s Theorem?

(3) (20 points) Let $f : \mathbb{R}^n \rightarrow [0, \infty]$ be a $\mathcal{B}$-measurable function which only takes non-negative values.

(a) Let $G_f = \{(x, y) \in \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R} : 0 \leq y < f(x)\}$ be the domain under the graph of $f$. Show that $G_f$ is a Borel set and $\lambda(G_f) = \int_{\mathbb{R}^n} f d\lambda$.

(b) Define $g : [0, \infty] \rightarrow [0, \infty]$ by $g(t) = \lambda(f^{-1}([t, \infty]))$ (you can also solve the problem for $g(t) = \lambda(f^{-1}((t, \infty)))$). Show that

$$\int_{\mathbb{R}^n} f \, d\lambda = \int_{[0,\infty)} g \, d\lambda = \int_{0}^{\infty} g(t) \, dt.$$ 

(c) (0 points, you don’t need to write up the answer for this question) How would you prove the identity in (b) if $f : X \rightarrow [0, \infty]$ is a general $\mathcal{M}$-measurable function on a space $X$?

(4) (20 points) Consider the radial distance function $r : \mathbb{R}^n \rightarrow \mathbb{R}$, $r(x) = |x|$.

(a) Show that for any $\mathcal{B}$-measurable function $f : [0, \infty) \rightarrow \mathbb{R}$ also the function $f \circ r : \mathbb{R}^n \rightarrow \mathbb{R}$ is $\mathcal{B}$-measurable and use the transformation law and Fubini’s
Theorem to show that there is a constant \( \omega_n > 0 \) (which only depends on \( n \)) such that

\[
\int_{\mathbb{R}^n} (f \circ r) \, dx = \omega_n \int_0^\infty f(y) y^{n-1} \, dy.
\]

Argue that \( \omega_1 = 2, \omega_2 = 2\pi \) and that for all \( R \geq 0 \) we have

\[
\lambda(B(0, R)) = \omega_n R^n.
\]

(Hint: You can either use a function which maps a halfspace \((0, \infty) \times \mathbb{R}^{n-1}\) to itself and transforms every hemisphere into a hyperplane, or you can use the fact that there is a diffeomorphism from \((\mathbb{R}^n \setminus \{0\}) \times (0, 1)\) to \(V \times (0, \infty)\) where \(V \subset \mathbb{R}^n\) is a neighborhood of the unit sphere. In dimension \( n = 2 \) you can use polar coordinates.)

(b) Use Fubini’s Theorem to show that

\[
\int_{\mathbb{R}^n} \exp(-|x|^2) \, d\lambda = \left( \int_{\mathbb{R}^2} \exp(-|x|^2) \, d\lambda \right) \left( \int_{\mathbb{R}^{n-2}} \exp(-|x|^2) \, d\lambda \right)
\]

and

\[
\int_{\mathbb{R}^2} \exp(-|x|^2) \, d\lambda = \left( \int_{\mathbb{R}} \exp(-|x|^2) \, d\lambda \right)^2.
\]

(c) Use part (a) and (b) to show that

\[
\int_{-\infty}^\infty \exp(-x^2) \, dx = \sqrt{\pi}.
\]

(d) Use part (a)-(c) to show the recursion formula

\[
\omega_n = \frac{2\pi}{n-2} \omega_{n-2}.
\]

Hence

\[
\omega_{2n} = \frac{(2\pi)^n}{(2n-2)(2n-4) \cdots 2}, \quad \omega_{2n-1} = \frac{(2\pi)^{n-1}}{(2n-3)(2n-5) \cdots 1}.
\]

(e) Let \( b \in \mathbb{C} \). Show that for any \( t \in \mathbb{R} \), the integral

\[
g(t) = \int_{-\infty}^\infty \exp(-x^2 + t bx) \, dx
\]

exists and show that \( g(t) \) is differentiable with \( g'(t) = -\frac{bt^2}{2} g(t) \). Conclude from this that for any real \( a > 0 \) and \( b_1, \ldots, b_n \in \mathbb{C} \), we have

\[
\int_{\mathbb{R}^n} \exp\left(-a|x|^2 + b_1 x_1 + \ldots + b_n x_n\right) \, d\lambda = \left( \frac{\pi}{a} \right)^{n/2} \exp\left(\frac{b_1^2 + \ldots + b_n^2}{4a}\right).
\]

(5) (20 points) Let \( f, \phi : \mathbb{R}^n \to \mathbb{R} \) be \( \mathcal{B} \)-measurable and integrable functions. Assume that \( \int \phi \, d\lambda = 1 \) and set \( \phi_a(x) = a^{-n} \phi\left(\frac{x}{a}\right) \) for all \( a > 0 \). In the following, we consider the convolution

\[
(f \ast \phi_a)(x) = \int f(x-y) \phi_a(y) \, dy = \int f(y) \phi_a(x-y) \, dy \quad \text{a.e.}
\]
(a) Show that

\[ \lim_{y \to 0} \int_{\mathbb{R}^n} |\phi(x + y) - \phi(x)|d\lambda(x) = 0. \]

(Hint: Show that this is true in the case in which \( \phi \) is continuously differentiable and has compact support. Then approximate \( \phi \) by a smooth function of compact support \( \phi^* \) such that \( \|\phi - \phi^*\|_1 \) is arbitrary small.)

(b) Show that if \( f \) or \( \phi \) is bounded, then \( f \ast \phi_a \) is continuous for any \( a > 0 \). Show moreover for general \( f, \phi \) and any \( m \geq 0 \): If \( \phi \) is in \( C^m_c \) (i.e. \( m \) times continuously differentiable of compact support), then \( f \ast \phi_a \) is in \( C^m \).

(c) Show that \( \|f \ast \phi_a\|_1 \leq \|f\|_1 \) for all \( a > 0 \).

(d) Assume that \( f \) is continuous and of compact support. Show that for all \( x \in \mathbb{R}^n \) we have \( \lim_{a \to 0} (f \ast \phi_a)(x) = f(x) \) and that \( \lim_{a \to 0} \|f - f \ast \phi_a\|_1 = 0 \).

(Hint: Show this first in the case in which \( \phi \) has compact support, then approximate \( \phi \).)

(e) Now assume that \( f \) is only \( \mathcal{B} \)-measurable and integrable. Show that still \( \lim_{a \to 0} \|f - f \ast \phi_a\|_1 = 0 \).

(Hint: Approximate \( f \) by a continuously differentiable function \( f^* \) of compact support such that \( \|f - f^*\|_1 \) is arbitrary small. Then use part (d) and (c).)

(f) Now consider a sequence \( \phi_1, \phi_2, \ldots : \mathbb{R}^n \to \mathbb{R} \) of \( \mathcal{B} \)-measurable functions such that

(i) \( \lim_{k \to \infty} \int \phi_k d\lambda = 1 \).

(ii) There is a constant \( C \) such that \( \int |\phi_k|d\lambda < C \) for all \( k \geq 1 \).

(iii) For any \( r > 0 \), we have \( \lim_{k \to \infty} \int_{\mathbb{R}^n \setminus B(0,r)} |\phi_k|d\lambda = 0 \).

Show that

\[ \lim_{k \to \infty} \|f - f \ast \phi_k\|_1 = 0. \]

(g) Prove the following: If \( f \in L^1(\mathbb{R}^n) \) has the property that \( \int fg d\lambda = 0 \) for any smooth function of compact support \( g \in C^\infty_c(\mathbb{R}^n) \), then \( f = 0 \) a.e.

(6) (0 points, you don't need to write up the solution to this problem) Let \( U \subset \mathbb{R}^n \) be open and consider a \( C^1 \)-map \( \Phi : U \to \mathbb{R}^n \) (\( \Phi \) does not need to be invertible). Denote by \( \text{Im} \Phi \subset \mathbb{R}^n \) its image.

(a) Assume first that \( d\Phi_x \) is invertible for all \( x \in U \). Show that there is a sequence of open sets \( U_1, U_2, \ldots \subset U \) and open sets \( V_1, V_2, \ldots \subset \mathbb{R}^n \) such that \( U = \bigcup_{k=1}^\infty U_k \) and such that \( \Phi \) restricted to \( U_k \) is a \( C^1 \)-diffeomorphism for each \( k \), i.e. \( \Phi_k = \Phi|_{U_k} : U_k \to V_k \) is a diffeomorphism.

(Hint: Use the Implicit Function Theorem.)

(b) Assume first that \( d\Phi_x \) is invertible for all \( x \in U \). Show that the function \( \#\Phi^{-1} : \mathbb{R}^n \to \mathbb{R} \), which assigns to every \( x \in \mathbb{R}^n \), the number of elements in the preimage \( \Phi^{-1}(x) \), is \( \mathcal{B} \)-measurable and for any \( \mathcal{B} \)-measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) we have

\[ \int_U (f \circ \Phi) |\det d\Phi|d\lambda = \int_{\mathbb{R}^n} f(\#\Phi^{-1})d\lambda \]

as long as one (and hence both) of the two expressions is integrable.

(Hint: Represent \( \#\Phi^{-1} \) as a linear combination of characteristic functions using the \( \Phi_k \).)
(c) Now drop the assumption that \( d\Phi \) is invertible everywhere. Let \( K \subset U \) be a compact subset. Show that there is a constant \( C \) such that the following holds: Let \( d > 0 \). Consider a special rectangle \( Q \subset K \) of which no side is more than twice as long as another. Then if \( |\det d\Phi_x| \leq d \) for all \( x \in Q \), we have

\[
\lambda^*(\Phi(Q)) \leq Cd\lambda(Q).
\]

(Maybe it’s easier to show \( \lambda^*(\Phi(Q)) \leq C\sqrt{d}\lambda(Q) \).)

(d) In the same setting as in (c) show that: If \( G \subset K \subset U \) is an open set such that \( |\det d\Phi_x| < d \) for all \( x \in G \), then

\[
\lambda^*(\Phi(G)) \leq Cd\lambda(G).
\]

(e) Show SARD’s Theorem: The set of all singular points of \( \Phi \) is a nullset. I.e. let \( A = \{x \in U : d\Phi \text{ is not invertible}\} \). Then \( \lambda(\Phi(A)) = 0 \).

(f) Conclude the general transformation law: For any \( \mathcal{B} \)-measurable function \( f : \mathbb{R}^n \to \mathbb{R} \) and continuously differentiable map \( \Phi : U \to \mathbb{R}^n \) we have

\[
\int_U (f \circ \Phi) |\det d\Phi|d\lambda = \int_{\mathbb{R}^n} f(\Phi^{-1})d\lambda
\]

as long as one (and then both) expressions are integrable.

*Maximum total points: 70*