Solution to Problem Set 8

(1) \( A = \{ x \in \mathbb{R}^n : f_1(x) \neq f_2(x) \} \subseteq \mathbb{R}^n \) is a nullset and open. If \( x \in A \), then \( B(x, r) \subseteq A \) for some \( r > 0 \).

\[ \Rightarrow 0 = \lambda(A) = \lambda(B(x, r)) > 0 \]

So \( A = \emptyset \)

(2) Obviously, \( f \) is \( B \)-meas. since \( f \) is continuous.

So \( f(\cdot, y) \) and \( f(x, \cdot) \) are \( B \)-meas. For any \( x \in (0,1) \)

\[ \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy \leq \int_0^1 \frac{1}{x^2 + y^2} \, dy \leq \int_0^1 \frac{1}{x^4} \, dx = \frac{1}{x^4} < \infty \]

For \( x \neq \frac{1}{2} \) so \( f(x, \cdot) \) is integrable.

Same for \( f(\cdot, x) = -f(x, \cdot) \). Now for \( x \in (0,1) \)

\[ \int_0^1 f(x, y) \, dy = \int_0^1 \frac{x^2 - y^2}{(x^2 + y^2)^2} \, dy = \frac{y}{x^2 + y^2} \bigg|_0^1 = \frac{1}{x^2 + 1} \]

\[ \int_0^1 (\int_0^1 f(x, y) \, dy) \, dx = \int_0^1 \frac{dx}{x^2 + 1} = \arctan(x) \bigg|_0^1 = \frac{\pi}{4} \]

\[ \int_0^1 (\int_0^1 f(x, y) \, dx) \, dy = -\int_0^1 (\int_0^1 f(y, x) \, dy) \, dx = -\frac{\pi}{4} \]

Fubini doesn't work here, because \( f \) is not integrable on \((0,1) \times (0,1)\). In fact

\[ \int_{(0,1) \times (0,1)} |f(x, y)| \, dx \, dy = \int_{(0,1) \times (0,1)} \frac{|x^2 - y^2|}{(x^2 + y^2)^2} \, dx \, dy \]

\[ \geq \int_D \frac{3}{4} \cdot \frac{x^2}{(\frac{3}{4} \cdot x^2)^2} \, dx \, dy = \frac{12}{25} \int_D \frac{1}{x^2} \, dx \, dy \]

\[ D = \{(x, y) \in (0,1) \times (0,1) : x < y \} \]

\[ = \frac{12}{25} \int_0^1 \left( \int_0^x \frac{1}{x^2} \, dy \right) \, dx = \frac{12}{25} \int_0^1 \frac{dx}{2x} = \infty \].
(3) Claim \[ \begin{align*} & A \in \mathbb{B}^n, A \subseteq \mathbb{R}^n, [a,b] \subseteq \mathbb{R}^n \Rightarrow \quad A \times [a,b] \in \mathbb{B}^{n+1} \\ & \text{Proof:} \quad \mathcal{M} = \{ A \in \mathbb{B}^n : A \times [a,b] \in \mathbb{B}^{n+1} \} \text{ is a C-algebra and } \mathcal{G}^n \subseteq \mathcal{M} \quad (\text{since } A \in \mathcal{G}^n \Rightarrow A \times (a-\varepsilon, b+\varepsilon) \in \mathcal{G}^{n+1} \text{ for all } \varepsilon > 0 \Rightarrow A \times [a,b] = \bigcap_{k=1}^{\infty} A \times [a-1/k, b+1/k] \in \mathbb{B}^{n+1}). \\ & \text{So } \mathcal{M} = \mathbb{B}^{n+1}. \quad \square 
\]

Claim 2 \[ \quad \begin{align*} & A \subseteq \mathbb{R}^n, A \in \mathbb{B}_0 \Rightarrow \quad \mathcal{G} \chi_A \in \mathbb{B}^{n+1} \\ & \text{Proof:} \quad \mathcal{G} \chi_A = A \times [a, b] \in \mathbb{B}^{n+1} \\ & \square 
\]

Claim 3 \[ \quad \begin{align*} & s : \mathbb{R}^n \to \mathbb{R} \text{ simple, } \mathcal{G}^n \text{-meas.}, \quad s > 0 \Rightarrow \quad \mathcal{G}_s \subseteq \mathbb{B}^{n+1} \\ & \text{Proof:} \quad s = a_1 \chi_{A_1} + \cdots + a_m \chi_{A_m} \\ & \text{Then} \quad \mathcal{G}_s = \bigcup \chi_{A_1} \times [0, a_1] \cup \cdots \cup \chi_{A_m} \times [0, a_m] \in \mathbb{B}^n \\ & = \mathcal{G}_{a_1} \chi_{A_1} \cup \cdots \cup \mathcal{G}_{a_m} \chi_{A_m} \quad \square 
\]

Claim 4 \[ \quad \begin{align*} & \mathcal{G}_f \subseteq \mathbb{B}^{n+1} \\ & \text{Proof:} \quad \text{Choose } s_1, s_2, \ldots, \lim_{k \to \infty} s_k = f \quad \text{simple} \\ & \text{Then} \quad \mathcal{G}_f = \mathcal{G}_{s_1} \cup \mathcal{G}_{s_2} \cup \cdots \subseteq \mathbb{B}^{n+1} \quad \square 
\]

Now consider the splitting \[ \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}. \text{ By Fubini} \]
\[ \int_{\mathbb{R}^{n+1}} \chi_{G_f} \, d\lambda = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}} \chi_{G_f}(x,y) \, dy \right) \, dx = \int_{\mathbb{R}^n} f(x) \, dx = f(x) \]

(see on next page)

(4) Consider the splitting \[ \mathbb{R}^{n+1} = \mathbb{R}^n \times \mathbb{R}. \text{ For } y > 0 \]
\[ \lambda(G_f(y)) = \lambda(f^{-1}((y, \infty))). \text{ By Fubini} \]
\[ \int_{\mathbb{R}^n} f \, dx = \int_{\mathbb{R}^{n+1}} \chi_{G_f} \, d\lambda = \int_{\mathbb{R}} \left( \int_{\mathbb{R}^n} \chi_{G_f}(x,y) \, dx \right) \, dy \]
\[ = \int_{\mathbb{R}} \lambda(G_f(y)) \, dy = \int_0^\infty \lambda(f^{-1}((y, \infty))) \, dy \]


If \( \lambda(f''((y,\infty])) = \infty \) for some \( y > 0 \), then
\[
\int_{\mathbb{R}^n} f \, d\lambda = \int_0^\infty g(t) \, dt = \infty
\]
and we're done.

So assume \( \lambda(f''((y,\infty])) < \infty \) for all \( y > 0 \).

Then \( g(t) = \lim_{y \to t^-} \lambda(f''((y,\infty])) \).

Since \( g \) is monotone, this value only differs from \( \lambda(f''((t,\infty]])) \)
at countably many points. So \( \lambda(f''((y,\infty])) = g(y) \)
for a.e. \( y \in [0,\infty) \). Hence
\[
\int f \, d\lambda = \int_0^\infty g(t) \, dt.
\]

(c) First check, that the claim is true for simple functions. For general \( f \), find \( s_1, s_2, \ldots \)
s.t. \( f = \lim_{k \to \infty} s_k \). Let \( g_k(t) = \frac{1}{\lambda(s_k([t,\infty]))} \).

Then \( f''((t,\infty]) = \bigcup_{k=1}^{\infty} s_k''((t,\infty]) \), so
\[
\lambda(f''((t,\infty])) = \lim_{k \to \infty} g_k(t).
\]
Since \( g_1 \leq g_2 \leq \ldots \),
\[
\int f \, d\lambda = \int_0^\infty g(t) \, dt \leq \int_0^\infty \lambda(f''((t,\infty])) \, dt
\]

**Alternative method to show that \( G_f \in B^{n+1} \).**

(due to Sam Pimentel)

1° We show as in Claim 1 that
\[
A \in B^n \Rightarrow A \times \mathbb{R} \in B^{n+1}.
\]
So \( \tilde{f}(x_1, \ldots, x_n, x_{n+1}) = f(x_1, \ldots, x_n) \), \( \tilde{f} : R^{n+1} \to \overline{\mathbb{R}} \)
is \( B^{n+1} \)-measurable (observe that \( \tilde{f}''(B) = f''(B) \times \mathbb{R} \)
for all \( B \in B(\overline{\mathbb{R}}) \).

2° Hence \( \hat{g} : R^{n+1} \to \overline{\mathbb{R}}, \hat{g}(x_1, \ldots, x_{n+1}) = \tilde{f}(x_1, \ldots, x_{n+1}) - x_{n+1} \)
is \( B^{n+1} \)-measurable.

3° Now \( G_f = \hat{g}((0,\infty]) \cap \overline{\mathbb{R} \times [0,\infty)} \in B^{n+1} \)
closed.
Method 1  Consider the smooth diffeo
\[ F : (0, \infty) \times \mathbb{R}^{n-1} \longrightarrow (0, \infty) \times \mathbb{R}^{n-1} \]
\[ x = (x_1, x_2, \ldots, x_n) \mapsto \frac{x_1}{|x^1|} \]

We can compute the Jacobian of \( F \) using the one indicated on the left:
\[ \det dF_x = \left( \frac{x_1}{|x^1|} \right)^{n-1} \frac{x_1}{|x^1|} = \left( \frac{x_1}{|x^1|} \right)^n \]

So by the Transformation law
\[
\int_{(0, \infty) \times \mathbb{R}^{n-1}} (\text{for } \lambda) \, d\lambda = \int_{(0, \infty) \times \mathbb{R}^{n-1}} (\text{for } F) \left( \frac{x_1}{|x^1|} \right)^n \, d\lambda
\]
\[ = \int_{(0, \infty) \times \mathbb{R}^{n-1}} f(x_1) \left( \frac{x_1}{|x^1|} \right)^n \, d\lambda = \int_0^\infty f(x_1) \left( \int_{\mathbb{R}^{n-1}} \left( \frac{x_1}{|x^1|} \right)^n \, dy_2 \ldots dy_n \right) \, dx_1
\]
\[ = \int_0^\infty f(x_1) \, x_1^{n-1} \, dx_1
\]

Analogously,
\[ \int_{(-\infty, 0) \times \mathbb{R}^{n-1}} (\text{for } \lambda) \, d\lambda = \frac{\omega_{n-1}}{2} \int_0^\infty f(x_1) x_1^{n-1} \, dx_1
\]
So since \( \{0\} \times \mathbb{R}^{n-1} \) is a nullset
\[
\int_{\mathbb{R}^{n-1}} (\text{for } \lambda) \, d\lambda = \int_{(-\infty, 0) \times \mathbb{R}^{n-1}} (\text{for } \lambda) \, d\lambda + \int_{(0, \infty) \times \mathbb{R}^{n-1}} (\text{for } \lambda) \, d\lambda = \omega_n \int_0^\infty f(y) y^{n-1} \, dy
\]
Method 2 Consider the smooth diffeo $F : D = (B(0,1) \setminus \{0\}) \times (0, \infty) \longrightarrow (\mathbb{R}^n \setminus \{0\}) \times (0, 1)$

$(\bar{x}, \ y) \longmapsto (y^{\frac{\bar{x}}{|\bar{x}|}}, |\bar{x}|)$

The Jacobian is $|\det dF(\bar{x}, \ y)| = (\frac{y}{|\bar{x}|})^{n-1}$

By Fubini and the Transformation law

$$\int_{\mathbb{R}^n} (f \circ o \bar{F}) \ d\lambda = \int_{\mathbb{R}^n \setminus \{0\}} (f \circ o \bar{F}) \ d\lambda \overset{\text{Fubini}}{=} \int (f \circ o \bar{F})(\bar{x}) \ d\bar{x} \ dy$$

$$\overset{\text{Transformation Law}}{=} \int_D (f \circ o \bar{F})(\frac{y}{|\bar{x}|})^{n-1} \ d\lambda$$

... since $(f \circ o \bar{F})(\bar{x}, \ y) = y$

$$\overset{\text{Transformation Law}}{=} \int_D f(y) (\frac{y}{|\bar{x}|})^{n-1} \ d\bar{x} \ dy$$

$$\overset{\text{Fubini}}{=} \int_0^\infty f(y) y^{n-1} \left( \int_{B(0,1) \setminus \{0\}} \frac{1}{|\bar{x}|^{n-1}} \ d\bar{x} \right) \ dy$$

$$= \omega_n \int_0^\infty f(y) y^{n-1}.$$  

In particular if $f = \chi_{[0,1]}$

$$\int_0^\infty \chi_{B(0,1)} \ d\lambda = \int (f \circ o \bar{F}) \ d\lambda = \int_0^\infty y^{n-1} \left( \int_{B(0,1) \setminus \{0\}} \frac{1}{|\bar{x}|^{n-1}} \ d\bar{x} \right) \ dy$$

$$= \frac{1}{n} \left( \int_{B(0,1) \setminus \{0\}} \frac{1}{|\bar{x}|^{n-1}} \ d\lambda \right) = \frac{\omega_n}{n}.$$  

We know that $\int_{\mathbb{R}^n} \chi_{B(0,1)} = \pi \Rightarrow \omega_2 = 2\pi$

$\int_{\mathbb{R}^1} \chi_{B(0,1)} = 2 \Rightarrow \omega_1 = 2$
(b) Fubini's Theorem for non-negative functions yields
\[
\int_{\mathbb{R}^n} \exp(-|x|^2) \, dx = \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^{n-2}} \exp\left(-\left(\sum_{i=1}^{n-2} x_i^2\right)\right) dx_{n-2} \right) \, dx_2
\]
\[
= \int_{\mathbb{R}^2} \left( \int_{\mathbb{R}^{n-2}} \exp\left(-\sum_{i=1}^{n-2} x_i^2\right) dx_{n-2} \right) \exp(-x_1^2) \, dx_2
\]
\[
= \left( \int_{\mathbb{R}^{n-2}} \exp(-|x|^2) \, dx \right) \left( \int_{\mathbb{R}^2} \exp(-|x|^2) \, dx \right)
\]

Same for \( \int_{\mathbb{R}^2} \exp(-|x|^2) \, dx \)

(c) \[
\int_{\mathbb{R}^2} \exp(-|x|^2) \, dx = \int_{\mathbb{R}^2} \exp(-r^2) \, d\lambda = 2\pi \int_0^\infty y \exp(y^2) \, dy
\]

Substitution \( t = y^2 \)
\[\pi \int_0^\infty \exp(-t) \, dt = \pi \]
\[\pi = \int_{\mathbb{R}^2} \exp(-|x|^2) \, d\lambda = \left( \int_{-\infty}^\infty \exp(-x^2) \, dx \right)^2
\]
So \( \int_{-\infty}^\infty \exp(-x^2) \, dx = \sqrt{\pi} \).

(d) By part (b)
\[
\int_{\mathbb{R}^n} \exp(-|x|^2) \, d\lambda = \pi \int_{\mathbb{R}^{n-2}} \exp(-|x|^2) \, d\lambda
\]
\[
\omega_n \int_0^\infty y^{n-1} \exp(-y^2) \, dy \quad \pi \omega_{n-2} \int_0^\infty y^{n-3} \exp(-y^2) \, d\lambda
\]
\[
\frac{2\pi \omega_{n-2}}{n-2} \int_0^\infty y^{n-1} \exp(-y^2) \, d\lambda
\]
Since \( \int_0^\infty y^{n-1} \exp(-y^2) \, dy < \infty \). We conclude

\[
\omega_n = \frac{2\pi}{n-2} \omega_{n-2}
\]

The formulas follow using \( \omega_1 = 2, \omega_2 = 2\pi \).

(e) The integral exists since \( \text{Re}(-x^2 + tbx) \leq C - \frac{x^2}{2} \).

For small \( h \in \mathbb{R} \) (i.e. \( |h| < 1 \))

\[
\frac{g(t+h) - g(t)}{h} = \int_0^\infty \exp(-x^2 + tbx) \frac{e^{hbx} - 1}{h} \, dx
\]

The absolute value of the integrand is bounded by

\[|\exp(-x^2 + tbx)| |b|x| (|\exp(bx)| + |\exp(-bx)|)\]

which is integrable. To deduce this inequality you can argue

\[
e^{hbx} - 1 = \int_0^1 \frac{d}{dt} (e^{tbx}) \, dt = hbx \int_0^1 e^{tbx} \, dt
\]

\[
\Rightarrow \left| \frac{e^{hbx} - 1}{h} \right| \leq |bx| \int_0^1 |e^{tbx}| \, dt
\]

By LDGCT

\[
g'(t) = \int_{-\infty}^{\infty} bx \exp(-x^2 + tbx) \, dx
\]

\[
= \left[ \frac{bx \exp(-x^2) (-\frac{b^2}{2} \exp(tbx))}{-2x \exp(-x^2)) (-\frac{b^2}{2} \exp(tbx))} \right]_{-\infty}^{\infty}
\]

So since \( g(0) = \sqrt{\pi} \), we have \( g(t) = \sqrt{\pi} \exp\left(-\frac{b^2}{4}t\right) \).

Hence \( \int_{-\infty}^{\infty} \exp(-x^2 + tbx) \, dx = g(1) = \sqrt{\pi} \exp\left(-\frac{b^2}{4}ight) \).

By substitution \( x = \sqrt{a} x \)

\[
\int_{-\infty}^{\infty} \exp(ax^2 + bx^2) \, dx = \sqrt{\frac{\pi}{a}} \exp(-\frac{b^2}{4a})
\]

The rest is Fubini.
If \( \phi \in C^1_c(\mathbb{R}^n) \), then \( \phi \) is Lipschitz and we can assume that \( \phi(x) = 0 \) if \( x \in \mathbb{R}^n \setminus B(0,r) \). Let \( L \) be the Lipschitz constant. Then

\[
\int_{\mathbb{R}^n} |\phi(x+y) - \phi(x)| \, dx \leq \int_{B(0,r+y)} L |y| \lambda(B(0,r+|y|)) \, dy \xrightarrow{|y| \to 0} 0
\]

Now assume that \( \phi \) is general. For every \( \varepsilon > 0 \) find \( \phi^* \in C^1_c(\mathbb{R}^n) \) s.t. \( \|\phi - \phi^*\|_1 < \varepsilon \) and choose \( \delta > 0 \) s.t. for all \( |y| < \delta \)

\[
\int_{\mathbb{R}^n} |\phi^*(x+y) - \phi^*(x)| \, dx < \varepsilon
\]

Then

\[
\begin{align*}
\int_{\mathbb{R}^n} |\phi(x+y) - \phi(x)| \, dx & \leq \int_{\mathbb{R}^n} |\phi(x+y) - \phi^*(x+y)| \, dx \\
& \quad + \int_{\mathbb{R}^n} |\phi^*(x+y) - \phi^*(x)| \, dx \\
& \quad + \int_{\mathbb{R}^n} |\phi^*(x) - \phi(x)| \, dx \\
& \leq 3\varepsilon
\end{align*}
\]

By symmetry, we can assume that \( |f| \leq M \). Then using (a) for any \( y \in \mathbb{R}^n \) we have

\[
| (f \ast \phi_a)(x+y) - (f \ast \phi_a)(x) | \\
\leq \left| \int_{\mathbb{R}^n} f(x-z) (\phi_a(z+y) - \phi_a(z)) \, dz \right| \\
\leq \int_{\mathbb{R}^n} |f(x-z)| |\phi_a(z+y) - \phi_a(z)| \, dz \\
\leq M \int |\phi_a(z+y) - \phi_a(z)| \, dz \xrightarrow{y \to 0} 0
\]
For the second part assume that $\phi \in C_c^1(\mathbb{R}^n)$, i.e. $\phi_a \in C_c^1(\mathbb{R}^n)$ too. Then for any $k = 1, \ldots, n$

\[
(f \ast \phi_a)(x + he_k) - (f \ast \phi_a)(x) = \int_{\mathbb{R}^n} f(x - y) \frac{\phi_a(y + he_k) - \phi_a(y)}{h} dy
\]

Since $\phi_a$ is Lipschitz, the integrand is dominated by $\|f(x - y)\| \cdot L$ which is integrable. So by LDCT

\[
\frac{\partial}{\partial x_k} (f \ast \phi_a)(x) = \int_{\mathbb{R}^n} \lim_{h \to 0} f(x - y) \frac{\phi_a(y + he_k) - \phi_a(y)}{h} dy
\]

\[
= \int_{\mathbb{R}^n} f(x - y) \frac{\partial \phi_a}{\partial x_k} (y) dy = f \ast \frac{\partial \phi_a}{\partial x_k}
\]

Hence $f \ast \phi_a \in C^1(\mathbb{R}^n)$. Now if $\phi_a \in C^m_c$, then $\frac{\partial \phi_a}{\partial x_k} \in C^{m-1}(\mathbb{R}^n)$ for all $k$.

\[
\text{induction } \frac{\partial}{\partial x_k} (f \ast \phi_a) \in C^{m-1}(\mathbb{R}^n) \text{ for all } k
\]

(c) $\|f \ast \phi_a\|_1 \leq \|f\|_1 \|\phi_a\|_1 = \|f\|_1 \|\phi\|_1 \leq C \|f\|_1$

(d) $(f \ast \phi_a)(x) = \int f(x - y) \phi_a(\frac{y}{a}) dy = \int f(x - ay) \phi(y) dy$

Since $f \in C_c^0(\mathbb{R}^n)$, we know that $|f| \leq M$. Hence the integrand is dominated by $M |\phi|$. By continuity

\[
\lim_{a \to 0} f(x - ay) = f(x).
\]

By LDCT

\[
\lim_{a \to 0} (f \ast \phi_a)(x) = \int f(x) \phi(y) dy = f(x) \cdot 1
\]

We now show the $L^1$-convergence. Assume first that $\phi$ has compact support, i.e. $\phi(x) = 0$ if $x \in \mathbb{R}^n \setminus B(0)$.

Recall also that $f(x) = 0$ for all $x \in \mathbb{R}^n \setminus B(0, R)$ for some $R > 0$. Then $(f \ast \phi_a)(x) = 0$ for all $x \in B(0, R)$. Hence $\|f \ast \phi_a\|_1 \leq M \chi_{B(0, R)}$.

By LDCT

\[
\lim_{a \to 0} \|f \ast \phi_a - f\|_1 = 0
\]
If \( \phi \) is general, we argue as follows. For every \( \varepsilon > 0 \), there is an \( r > 0 \) s.t. \( \int_{\mathbb{R}^n \setminus B(0, r)} |\phi(x)| \, dx < \varepsilon \). Set \( \phi' = \phi \chi_{B(0, r)} \) and \( \phi'_a = a^{-n} \phi'(ax) \). Then applying the previous conclusion to the family \((\int \phi'(x) \, dx)\) \( \phi'_a \) yields \( \| f \ast (\int \phi'(x) \, dx) \phi'_a - f \|_1 \xrightarrow{a \to 0} 0 \). So \( \| f \ast \phi'_a - (\int \phi'(x) \, dx) \|_1 \xrightarrow{a \to 0} 0 \). So for a suff small \( \| f \ast \phi'_a - (\int \phi'(x) \, dx) \|_1 < \varepsilon \) and thus \( \| f \ast \phi'_a - f \|_1 < \varepsilon + \varepsilon < \varepsilon + 2\varepsilon \). The claim follows by letting \( \varepsilon \to 0 \).

(e) Given \( \varepsilon > 0 \) find \( f^* \in C_c^0(\mathbb{R}^n) \) s.t. \( \| f - f^* \|_1 < \varepsilon \). Then for suff. small \( a \) \( \| f^* \ast \phi_a - f \|_1 < \varepsilon \) and thus \( \| f \ast \phi_a - f \|_1 < \varepsilon + \varepsilon \). Again, assume first that \( f \in C_c^0(\mathbb{R}^n) \). Let \( \varepsilon > 0 \). By uniform continuity, there is a \( \delta > 0 \) s.t. \( |x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \varepsilon \). Choose \( k_0 \) s.t. for all \( k \geq k_0 \), \( \int_{\mathbb{R}^n \setminus B(0, \delta)} |\phi_k| \, dx < \varepsilon \). Then

\[
\left| (f \ast (\phi_k \chi_{B(0, \delta)}))(x) - \left( \int_{B(0, \delta)} \phi_k \, dy \right) f(x) \right|
\]
\[
\int \int (f(x-y) - f(y)) \chi_k(y) \chi_B(0, \delta)(y) \, dy \\
\leq \int_{B(0, \delta)} |f(x-y)-f(y)| \chi_k(y) \, dy \\
\leq C \int_{B(0, \delta)} |\chi_k| \, dy \\
= C \delta
\]

Assume that \( f(x) = 0 \) for all \( x \in \mathbb{R}^n \setminus B(0, R) \) and \( \delta < 1 \).
Then for all \( x \in \mathbb{R}^n \setminus B(0, R+1) \) we have \( (f \ast (\chi_k \chi_B(0, \delta))) (x) = 0 \).
Combining the previous two results yields
\[
\| f \ast (\chi_k \chi_B(0, \delta)) - (\int_{B(0, \delta)} \chi_k \, dy) \|_1 \\
\leq C \lambda \left( B(0, R+1) \right)
\]
Also
\[
\| f \ast (\chi_k \chi_{\mathbb{R}^n \setminus B(0, \delta)}) \|_1 \\
\leq \| f \|_1 \| \chi_k \chi_{\mathbb{R}^n \setminus B(0, \delta)} \|_1 \\
= \int_{\mathbb{R}^n \setminus B(0, \delta)} |\chi_k| \, dx < \epsilon
\]
So
\[
\| f \ast \chi_k - f \|_1 \\
\leq \| f \ast \chi_k - f \ast (\chi_k \chi_B(0, \delta)) \|_1 + \| f \ast (\chi_k \chi_B(0, \delta)) \\
- (\int_{B(0, \delta)} \chi_k \, dx) \|_1
\]
\[
\leq C \lambda \left( B(0, R+1) \right) + 2 \epsilon \| f \|_1 + \epsilon
\]
Letting \( \epsilon \to 0 \) yields the claim.

For general \( f \) and given \( \epsilon > 0 \) find \( f^* \in C_0^\infty (\mathbb{R}^n) \) s.t. \( \| f - f^* \|_1 < \epsilon \). Then there is a \( k_0 \) s.t. for all \( K > k_0 \)
\[
\| f - f^* \|_1 < \epsilon \\
\Rightarrow \| f \ast \chi_k - f \|_1 < \| f \ast \chi_k - f^* \ast \chi_k \|
\]
\[
+ \| f^* \ast \chi_k - f^* \|_1 + \| f^* - f \|_1
\]
\[
< 2 \epsilon + \| f - f^* \|_1 \| \chi_k \|_1 < 2 \epsilon + C \epsilon
\]
(g) Choose some function $\varphi \in C_c^\infty(\mathbb{R}^n)$ of integral 1, e.g. $(\int \varphi''(x)dx)^{-1} \varphi'$ for some $\varphi' \in C_c^\infty(\mathbb{R}^n)$ of nonzero integral. Then $f \ast \varphi_a = 0$ for all $a$.
Hence $\|f\|_4 = \|f-f \ast \varphi_a\|_4 \xrightarrow{a \to 0} 0 \Rightarrow \|f\|_4 = 0$.

Proof of the inequality $|\frac{e^{hb^x} - 1}{h} | \leq 1 + |e^{bx}|$ if $|h| \ll 1$.

$e^{hb^x} - 1 = e^{hb^x} - e^{hb^x} = \int_0^1 \frac{d}{dt} e^{hb^xt} dt$

$= hxb \int_0^1 e^{hb^x t} dt$

Since $|e^{hb^x t}| \leq e^{b|x|} \leq 1 + |e^{bx}|$. 
(6) (a) Let $x \in U$. By the implicit function theorem (or inverse function theorem), there is an open neighborhood $U_x \subset U$ around $x$ and an open neighborhood $V_x \subset \mathbb{R}^n$ around $\phi(x)$ s.t.

$$\phi|_{U_x} : U_x \rightarrow V_x \quad \text{(i.e. } \phi(U_x) = V_x)$$

is a diffeomorphism.

We have $U = \bigcup_{x \in U} U_x$. The claim follows by passing to a countable subcover.

(Recall: we can find a sequence of compact sets $K_1 \subset K_2 \subset \ldots$ s.t. $U = \bigcup_{i=1}^{\infty} K_i$. For each $K_i$ we can find a finite subcover)

(b) Define $A_1 = U_1 \setminus A_2 = U_2 \setminus U_1 \setminus \cdots$, $A_k = U_k \setminus \bigcup_{i=1}^{k-1} U_i$. Then $U = A_1 \cup A_2 \cup \cdots$. $A_k \in \mathcal{B}$. Set $B_k = \phi^{-1}(A_k) = (\phi^{-1})^{-1}(A_k) \in \mathcal{B}$.

Observe that $B_k \subset V_k$. Then for $x \in \mathbb{R}^n$

$$\# \phi^{-1}(x) = \sum_{k=1}^{\infty} \#(\phi^{-1}(x) \cap A_k) = \sum_{k=1}^{\infty} \#(x \cap B_k)$$

So

$$\# \phi^{-1} = \chi_{B_1} + \chi_{B_2} + \cdots$$

Hence $\# \phi^{-1}$ is $\mathcal{B}$-measurable. Moreover,

$$\int_U (f \circ \phi) |\det d\phi| \, d\lambda = \sum_{k=1}^{\infty} \int_{U_k} (f \circ \phi) \chi_{A_k} |\det d\phi| \, d\lambda$$

$$= \sum_{k=1}^{\infty} \int_{U_k} ((f \chi_{B_k}) \circ \phi_k) |\det d\phi_k| \, d\lambda$$

$$= \sum_{k=1}^{\infty} \int_{V_k} f \chi_{B_k} \, d\lambda = \int_{\mathbb{R}^n} f \left( \sum_{k=1}^{\infty} \chi_{B_k} \right) \, d\lambda$$

$$= \int_{\mathbb{R}^n} f (\# \phi^{-1}) \, d\lambda$$
(c) We carry out a modified version of the proof of the Lemma that was used for the Transformation Law:

If \( \det d\phi \neq 0 \) everywhere on \( Q \), then \( d\phi \) is invertible on \( Q \). This is then also true on an open neighborhood of \( Q \) and hence we can use part (b) to show the claim.

Assume now that \( \det d\phi_y = 0 \) for some \( y \in Q \).

First consider the case in which \( \text{diam } Q < \varepsilon \) where \( \varepsilon > 0 \) is chosen s.t. \( \| d\phi_{x_1} - d\phi_{x_2} \| < \varepsilon \) for all \( x_1, x_2 \in K \) (this is possible by uniform continuity).

Let \( S = d\phi_y \).

Then for any \( x \in Q \) and \( \sigma(t) = tx + (1-t)y \)

\[
\phi(x) - \phi(y) = \int_0^1 \frac{d}{dt} \phi(\sigma(t)) \, dt = \int_0^1 d\phi(\sigma(t))(x-y) \, dt
\]

Let \( L : \mathbb{R}^n \rightarrow \mathbb{R}^n \)

\( L(x) = S(x-y) + \phi(y) \)

\[
| \phi(x) - L(x) | = | \phi(x) - \phi(y) | - | L(x) - L(y) |
\]

\[
= \left| \int_0^1 \left( d\phi_{\sigma(t)} - d\phi_y \right)(x-y) \, dt \right|
\]

\[
\leq \int_0^1 \| d\phi_{\sigma(t)} - d\phi_y \| \| x-y \| \, dt \leq d \| x-y \| \ (\star)
\]

We can assume that \( \| d\phi \| \leq C' \) on \( K \).

Then since \( L \) is not invertible, \( L(Q) \) is contained in a hyperplane and has diameter \( \leq C' \text{diam } Q \).

By \((\star)\), \( \phi(Q) \) is contained in a \( d \text{diam } Q \) neighborhood of \( L(Q) \).
So \( \phi(Q) \) is contained in a (non-axes-parallel) rectangle of side lengths \((C^t + d) \text{diam}(Q), \ldots, (C^t + d) \text{diam}(Q)\), \(2d \text{ diam } Q \). So

\[
\lambda^*(\phi(Q)) \leq 2d (C^t + d)^{-\frac{n-1}{n}} \text{diam } Q
\]

which is reasonable assuming \( d < 1 \).

Now, for \( Q \) of general diameter, the claim follows by subdivision.

(d) The claim follows by expressing \( G \) as a countable disjoint union of special rectangles.

(e) Observe that \( A = \{ x \in U : \det d\phi_x = 0 \} \).

Let \( K_1 \subset K_2 \subset \cdots \) be a sequence of compact sets such that \( U = \bigcup_{i=1}^{\infty} K_i \) (interior of \( K_i \)).

Consider \( B_k = \{ x \in U : |\det d\phi_x| < \frac{1}{k} \} \). So \( A \subset B_k \).

Then by (d) we have for any fixed \( i \)

\[
\lambda^*(\phi(B_k \cap K_i)) \leq C_i \frac{1}{k} \lambda(B_k \cap K_i) \leq C_i \frac{1}{k} \lambda(K_i)
\]

\[
\Rightarrow \lambda^*(\phi(A \cap K_i)) \leq C_i \frac{1}{k} \lambda(K_i)
\]

\[
\Rightarrow \lambda^*(\phi(A \cap K_i)) = 0
\]

So \( \lambda(\phi(A)) = \lambda(\bigcup_{i=1}^{\infty} \phi(A \cap K_i)) = 0 \).

(f) Set \( A \subset U \) as before and consider \( U' = U \setminus A \).

Then \( U' \) is open and we can apply (b)

\[
\int_{U'} (f \circ \phi) |\det d\phi| = \int_{U} (f \circ \phi) |\det d\phi| \quad \text{(b)} \int_{\mathbb{R}^n} f(\#(\phi|_U)^{-1}) d\lambda
\]

since \( \#(\phi|_U)^{-1} \) and \( \#\phi^{-1} \) only differ on \( \phi(A) \) which is a nullset.

\[
= \int_{\mathbb{R}^n} f(\#\phi^{-1}) d\lambda
\]