MATH 173: PROBLEM SET 8
DUE 10AM, FRIDAY, MARCH 4, 2016

Problem 1. Solve the inhomogeneous heat equation on the half-line for Dirichlet boundary conditions:

$$u_t - ku_{xx} = f,\ u(x,0) = \phi(x),\ u(0,t) = 0,$$

in two different ways:

(i) Using Duhamel's principle, applied on the half line directly, and the solution formula for the homogeneous equation derived in class/lecture notes (i.e. with $f = 0$) on the half line.

(ii) Using the appropriate extension of f and ϕ to the whole real line and solving the inhomogeneous PDE on the real line.

Problem 2. Derive Duhamel's principle for the wave equation on $\mathbb{R}$

$$u_{tt} - c^2\partial_x^2 u = f,\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

by setting up a first order system for $U = \begin{bmatrix} u \\ v \end{bmatrix}$, $v = u_t$, namely

$$u_t - v = 0,\ u(x,0) = \phi(x),$$
$$v_t - c^2\partial_x^2 u = f,\ v(x,0) = \psi(x).$$

Thus, one has

$$\partial_t U - AU = \begin{bmatrix} 0 \\ f \end{bmatrix},\ U(0,x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & \mathrm{Id} \\ c^2\partial_x^2 & 0 \end{bmatrix}.$$

This is now a first order equation in time, so Duhamel's principle for first order equations is applicable, and gives the solution of the inhomogeneous equation as

$$U(x,t) = \mathcal{S}(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix} (x) + \int_0^t \mathcal{S}(t-s) \begin{bmatrix} 0 \\ f_s \end{bmatrix} (x)\, ds,$$

where $\mathcal{S}$ is the solution operator for the homogeneous problem $\partial_t U - AU = 0$. You need to work this out explicitly, in particular what $\mathcal{S}$ is, to derive the solution of the wave equation.

Problem 3. (i) Consider the following eigenvalue problem on $[0,\ell]$:

$$-X'' = \lambda X,\ X(0) = 0,\ X'(\ell) = 0.$$

Find all eigenvalues and eigenfunctions.

(ii) Using separation of variables, find the general 'separated' solution of the wave equation

$$u_{tt} = c^2 u_{xx},\ u(0,t) = 0,\ u_x(\ell,t) = 0.$$

(iii) Solve the wave equation with initial conditions

$$u(x,0) = \sin(3\pi x/(2\ell)) - 2\sin(5\pi x/(2\ell)),\ u_t(x,0) = 0.$$

(iv) Using separation of variables, find the general 'separated' solution of the *heat equation*

$$u_t = ku_{xx},\ u(0,t) = 0,\ u_x(\ell,t) = 0,$$

here $k > 0$ constant.

Problem 4. Consider the wave equation on a ring of length 2ℓ. We let x be the arclength variable along the ring, $x \in [-\ell, \ell]$. We would like to understand wave propagation along the ring, so consider the wave equation with *periodic boundary conditions*:

$$u_{tt} = c^2 u_{xx},\ u(-\ell, t) = u(\ell, t),\ u_x(-\ell, t) = u_x(\ell, t).$$

(i) Find the general 'separated' solution.

(ii) Find the solution with initial condition

$$u(x, 0) = 0,\ u_t(x, 0) = \cos(2\pi x/\ell) - \sin(\pi x/\ell),\ x \in [-\ell, \ell].$$

(iii) Give an alternative method of solution by extending u to be a 2ℓ-periodic function in x on all of $\mathbb{R}$, and using d'Alembert's formula.

(iv) How do singularities of u propagate? That is, if the only singularity of the initial data is at some x_0 (i.e. they are C^∞ elsewhere), where can u be singular? Interpret this physically.

Problem 5. The goal of this problem is to show that if $u \in \mathcal{D}'(\mathbb{R}^3)$ and $\Delta u = f$ satisfies $x_0 \notin \operatorname{singsupp} f$, i.e. f is C^∞ near x_0, then u is C^∞ near x_0. This is called elliptic regularity: Δ is elliptic, and for an elliptic operator P if Pu is C^∞ near some x_0 then so is u.

We achieve this as follows.

(i) First suppose that u is a C^2 function. Let $\phi \in C_c^\infty(\mathbb{R}^3)$ be identically 1 near x_0 such that f is C^∞ on $\operatorname{supp} \phi$. Then show that $\Delta(\phi u) = \phi \Delta u + v$, where v is a compactly supported distribution that vanishes near x_0. Now as $w = \phi u$ is compactly supported,

$$w(x) = -\int_{\mathbb{R}^3} \frac{1}{4\pi|x - y|} \Delta_y(\phi(y) u(y))\, dy.$$

(ii) Expand $\Delta_y(\phi(y)u(y))$ as above. To analyze

$$\int_{\mathbb{R}^3} \frac{1}{4\pi|x - y|} v(y)\, dy$$

for x near x_0, note that if x is near x_0 and $y \in \operatorname{supp} v$ then $x \neq y$, so $|x - y|^{-1}$ is C^∞. On the other hand, $\phi \Delta u$ is C^∞ by assumption. Write the corresponding part of the convolution as

$$\int_{\mathbb{R}^3} \frac{1}{4\pi|y|} \phi(x - y)(\Delta u)(x - y)\, dy,$$

and deduce that it is C^∞.

(iii) Suppose now that $u \in \mathcal{D}'$. Proceed as above, writing

$$w = -\frac{1}{4\pi|x|} * (\Delta(\phi u)),$$

convolution in the sense of distributions (so w is merely a distribution), and show that both parts are C^∞ near x_0. You do not have to be very careful in writing up this part; there are some technicalities, but the point is to get the main idea.