1 Borel Regular Measures

Recall that a Borel measure on a topological space $X$ is a measure defined on the collection of Borel sets, and an outer measure $\mu$ on $X$ is said to be a Borel-regular outer measure if all Borel sets are $\mu$-measurable and if for each subset $A \subset X$ there is a Borel set $B \supset A$ such that $\mu(B) = \mu(A)$. (Notice that this does not imply $\mu(B \setminus A) = 0$ unless $A$ is $\mu$-measurable and $\mu(A) < \infty$.) Also if $\nu$ is a Borel measure on $X$, then we get a Borel regular outer measure $\mu$ on $X$ by defining $\mu(Y) = \inf \nu(B)$ where the inf is taken over all Borel sets $B$ with $B \supset Y$, and this outer measure $\mu$ coincides with $\nu$ on all the Borel sets. (See Q.1 of hw2.)

Also, if $\mu$ is a Borel regular outer measure on $X$ and if $A \subset X$ is $\mu$-measurable with $\mu(A) < \infty$, then we claim $\mu \upharpoonright A$ is also Borel regular. Here $\mu \upharpoonright A$ is the outer measure on $X$ defined by

$$(\mu \upharpoonright A)(Y) = \mu(A \cap Y).$$

To check this claim first observe that if $E$ is $\mu$-measurable and $Y \subset X$ is arbitrary then $(\mu \upharpoonright A)(Y) = \mu(A \cap Y) = \mu(A \cap Y \cap E) + \mu(A \cap Y \setminus E) = (\mu \upharpoonright A)(Y \cap E) + (\mu \upharpoonright A)(Y \setminus E)$, hence $E$ is also $(\mu \upharpoonright A)$-measurable. In particular all Borel sets are $(\mu \upharpoonright A)$-measurable, so it remains to prove that for each $Y \subset X$ there is a Borel set $B \supset Y$ with $(\mu \upharpoonright A)(B) = (\mu \upharpoonright A)(Y)$. To prove this, first use the Borel regularity of $\mu$ and the fact that $A$ is measurable of finite measure to pick Borel sets $B_1 \supset Y \cap A$ and $B_2 \supset A$ with $\mu(B_1) = \mu(Y \cap A)$ and $\mu(B_2 \setminus A) = 0$, and then pick a Borel set $B_3 \supset B_2 \setminus A$ with $\mu(B_3) = 0$. Then $Y \subset B_1 \cup (X \setminus A) \subset B_1 \cup (X \setminus B_2) \cup B_3$ (which is a Borel set) and $(\mu \upharpoonright A)(Y) \leq (\mu \upharpoonright A)(B_1 \cup (X \setminus B_2) \cup B_3) = \mu((A \cap B_1) \cup (A \cap B_3)) \leq \mu(B_1) = \mu(Y \cap A) = (\mu \upharpoonright A)(Y)$.

We now state and prove an important regularity property of Borel regular outer measures:

1.1 Theorem. Suppose $X$ is a topological space with the property that every closed subset of $X$ is the countable intersection of open sets (this trivially holds e.g. if $X$ is a metric space), suppose $\mu$ is a Borel-regular outer measure on $X$, and suppose that $X = \bigcup_{j=1}^{\infty} V_j$, where $\mu(V_j) < \infty$ and $V_j$ is open for each $j = 1, 2, \ldots$. Then

$$\mu(A) = \inf_{U \text{ open}, U \supset A} \mu(U)$$

for each subset $A \subset X$, and

$$\mu(A) = \sup_{C \text{ closed}, C \subset A} \mu(C)$$

for each $\mu$-measurable subset $A \subset X$.

1.2 Remark: In case $X$ is a locally compact separable metric space (thus for each $x \in X$ there is $\rho > 0$ such that the closed ball $B_{\rho}(x) = \{ y \in X : d(x, y) \leq \rho \}$ is compact, and $X$ has a countable dense subset), the condition $X = \bigcup_{j=1}^{\infty} V_j$ with $V_j$ open and $\mu(V_j) < \infty$ is automatically satisfied provided $\mu(K) < \infty$ for each compact $K$. Furthermore in this case we have from (2) above that

$$\mu(A) = \sup_{K \text{ compact}, K \subset A} \mu(K)$$

for each $\mu$-measurable subset $A \subset K$ with $\mu(A) < \infty$, because under the above conditions on $X$ any closed set $C$ can be written $C = \bigcup_{i=1}^{\infty} K_i$, $K_i$ compact.

Proof of 1.1. We assume first that $\mu(X) < \infty$. By Borel regularity of $\mu$, for any given $A \subset X$ we can select a Borel set $B \supset A$ with $\mu(B) = \mu(A)$, so it clearly suffices to check (1) in the special case when $A$ is a Borel set. Now let

$$A = \{ \text{Borel sets } A : (1) \text{ holds} \}.$$

Trivially $A$ contains all open sets and one readily checks that $A$ is closed under both countable unions and intersections, as follows:

If $A_1, A_2, \ldots \in A$ then for any given $\varepsilon > 0$ there are open $U_1, U_2, \ldots$ with $U_j \supset A_j$ and $\mu(U_j \setminus A_j) \leq 2^{-j} \varepsilon$. Now one easily checks $U_j \cap (\bigcup_{K=k}^{\infty} A_k) \subset \bigcup_{j=1}^{\infty} (U_j \cap A_k)$ and $\bigcap_{j=1}^{\infty} U_j \cap (\bigcap_{k}^{\infty} A_k) \subset \bigcup_{j=1}^{\infty} (U_j \cap A_k)$ so by subadditivity we have $\mu(\bigcup_{j=1}^{\infty} (U_j \setminus (\bigcup_{k=1}^{\infty} A_k))) < \varepsilon$ and $\lim_{N \to \infty} \mu(\bigcap_{j=1}^{N} U_j \setminus (\bigcap_{k=1}^{N} A_k)) = \mu(\bigcup_{j=1}^{\infty} (U_j \setminus (\bigcap_{k=1}^{\infty} A_k))) < \varepsilon$, so both $U_k A_k$ and $\cap_k A_k$ are in $A$ as claimed.

In particular $A$ must also contain the closed sets, because any closed set in $X$ can be written as a countable intersection of open sets and all the open sets
are in $A$. Thus if we let $\tilde{A} = \{ A \in A : X \setminus A \in A \}$ then $\tilde{A}$ is a $\sigma$-algebra containing all the closed sets, and hence $\tilde{A}$ contains all the Borel sets. Thus $A$ contains all the Borel sets and (1) is proved in case $\mu(X) < \infty$.

To check (2) in case $\mu(X) < \infty$ we can just apply (1) to $X \setminus A$: thus for each $i = 1, 2, \ldots$ there is an open set $U_i \supset X \setminus A$ with $\mu(X) = \mu(A) + 1/i = \mu(X \setminus A) + 1/i > \mu(U_i) = \mu(X) - \mu(C_i)$, where $C_i = X \setminus U_i \subset X \setminus A = A$. Thus $C_i \subset A$ is closed and $\mu(C_i) > \mu(A) - 1/i$.

In case $\mu(X) = \infty$, to prove (1) we first take a Borel set $B$ with $\mu(B) = \mu(A)$ and then apply the above result for finite measures to the Borel regular measure $\mu \ll \nu$, giving closed sets $\{ D_n \}$ for each $n = 1, 2, \ldots$, such that $\mu(D_n) < 2^{-n}$ and hence $\mu(B \setminus \bigcup D_n) < 2^{-n}$.

Using this we prove Lusin’s Theorem:

**1.3 Theorem (Lusin’s Theorem.)** Let $\mu$ be a Borel regular outer measure on a topological space $X$ having the property that every closed set can be expressed as the countable intersection of open sets (e.g. $X$ is a metric space), let $A$ be $\mu$-measurable with $\mu(A) < \infty$, and let $f : A \to \mathbb{R}$ be $\mu$-measurable. Then for each $\varepsilon > 0$ there is a closed set $C \subset X$ with $C \subset A$, $\mu(A \setminus C) < \varepsilon$, and $f|C$ continuous.

**Proof:** For each $i = 1, 2, \ldots$, and $j = 0, \pm 1, \pm 2, \ldots$ let

$$A_{ij} = f^{-1}(i(j - 1)/i, j/i),$$

so that $A_{ij} \cup A_{i,j} \subset A_{ij}$. By the remarks preceding Theorem 1.1 we know that $\mu \ll A$ is a Borel regular outer measure and since it is finite we can apply Theorem 1.1 to it, and hence for given $\varepsilon > 0$ there is a closed set $C_{ij}$ in $X$ with $C_{ij} \subset A_{ij}$ such that $\mu(A_{ij} \setminus C_{ij}) = \mu(A_{ij} \setminus C_{ij}) < 2^{-i(j - 1)/i + 1},$ hence $\mu(A_{ij} \setminus (\bigcup_{i=\infty}^\infty C_{i,j})) < 2^{-i(j - 1)/i + 1} \varepsilon$ and hence

$$\mu(A \setminus (\bigcup_{i=\infty}^\infty C_{i,j})) < 2^{-i} \varepsilon.$$

so for each $i = 1, 2, \ldots$ there is a positive integer $j(i)$ such that $\mu(A \setminus (\bigcup_{i=1}^j C_{i,j})) < 2^{-i} \varepsilon$. Since $A \setminus (\bigcap_{i=1}^j (\bigcup_{i=1}^j C_{i,j})) = \bigcup_{i=1}^\infty (A \setminus (\bigcup_{i=1}^j C_{i,j}))$ (by De Morgan), this implies

$$\mu(A \setminus C) < \varepsilon,$$

where $C = \bigcap_{i=1}^\infty (\bigcup_{i=1}^j C_{i,j})$ is a closed subset of $A$.

Now define $g_i : \bigcup_{i=1}^\infty (\bigcup_{i=1}^j C_{i,j}) \to \mathbb{R}$ by setting $g_i(x) = (j(i) - 1)/i$ on $C_{i,j}$, $|j(i) - 1| \leq j(i)$. Then $g_i$ is clearly continuous and its restriction to $C$ is continuous for each $i$, furthermore by construction $0 \leq f(x) - g_i(x) \leq 1/i$ for each $x \in C$ and each $i = 1, 2, \ldots$, so that $g_i|C$ converges uniformly to $f|C$ on $C$ and hence $f|C$ is continuous. $\square$

## 2 Radon Measures, Representation Theorem

In this section we work mainly in locally compact Hausdorff spaces, and for the reader’s convenience we recall some basic definitions and preliminary topological results for such spaces.

Recall that a topological space is said to be Hausdorff if it has the property that for every pair of distinct points $x, y \in X$ there are open sets $U, V$ with $x \in U$, $y \in V$ and $U \cap V = \emptyset$. In such a space all compact sets are automatically closed, the proof of which is as follows: observe that if $x \notin K$ then for each $y \in K$ we can (by definition of Hausdorff space) pick open $U_y$, $V_y$ with $x \in U_y$, $y \in V_y$ and $U_y \cap V_y = \emptyset$. By compactness of $K$ there is a finite set $y_1, \ldots, y_N \in K$ with $K \subset \bigcup_{i=1}^N V_{y_i}$. Then $\bigcap_{i=1}^N U_{y_i}$ is an open set containing $x$ which is disjoint from $U_y \cap V_y$ and hence disjoint from $K$, so that $K$ is closed as claimed. In fact we proved a bit more: that for each $x \notin K$ there are disjoint open sets $U, V$ with $x \in U$ and $K \subset V$. Then if $L$ is another compact set disjoint from $K$ we can repeat this for each $x \in L$ thus obtaining disjoint open $U_x, V_x$ with $x \in U_x$ and $K \subset V_x$, and then compactness of $L$ implies $3x_1, \ldots, x_M \in L$ such that $L \subset \bigcup_{i=1}^M U_{x_i}$ and then $\bigcup_{i=1}^M U_{x_i}$ and $\bigcup_{i=1}^M V_{y_i}$ are disjoint open sets containing $L$ and $K$ respectively. By a simple inductive argument (left as an exercise) we can extend this to finite pairwise disjoint unions of compact subsets:

**2.1 Lemma.** Let $X$ be a Hausdorff space and $K_1, \ldots, K_N$ be pairwise disjoint compact subsets of $X$. Then there are pairwise disjoint open subsets $U_1, \ldots, U_N$
with \( K_j \subset U_j \) for each \( j = 1, \ldots, N \).

Notice in particular that we have the following corollary of Lemma 2.1:

### 2.2 Corollary

A compact Hausdorff space is normal: i.e. given closed disjoint subsets \( K_1, K_2 \) of a compact Hausdorff space, we can find disjoint open \( U_1, U_2 \) with \( K_j \subset U_j \) for \( j = 1, 2 \).

Most of the rest of the discussion here takes place in locally compact Hausdorff space: A space \( X \) is said to be locally compact if for each \( x \in X \) there is a neighborhood \( U_x \) of \( x \) such that the closure \( \overline{U}_x \) of \( U_x \) is compact.

An important preliminary lemma in such spaces is:

### 2.3 Lemma

If \( X \) is a locally compact Hausdorff space and \( V \) is a nbd. of a point \( x \), then there is a nbd. \( U_x \) of \( x \) such that \( \overline{U}_x \) is a compact subset of \( V \).

**Proof:** First pick a neighborhood \( U_0 \) of \( x \) such that \( \overline{U}_0 \) is compact and define \( W = U_0 \cap V \). Then \( \overline{W} \) is compact and hence so is the closed subset \( \overline{W} \setminus W \). Then \( \overline{W} \setminus W \) and \( \{x\} \) are disjoint compact sets so by Lemma 2.1 we can find disjoint open \( U_1, U_2 \) with \( x \in U_1 \) and \( \overline{W} \setminus W \subset U_2 \). Without loss of generality we can assume \( U_1 \subset W \) (otherwise replace \( U_1 \) by \( U_1 \cap W \)). Then \( \overline{U}_1 \subset X \setminus U_2 \subset X \setminus (\overline{W} \setminus W) \) and hence \( \overline{U}_1 \subset W \). Thus the lemma is proved with \( U_x = U_1 \).

**Remark:** In locally compact Hausdorff space, using Lemmas 2.1 and 2.3 it is easy to check that we can select the \( U_j \) in Lemma 2.1 above to have compact pairwise disjoint closures.

The following lemma is a version of the Urysohn lemma valid in locally compact Hausdorff space:

### 2.4 Lemma

Let \( X \) be a locally compact Hausdorff space, \( K \subset X \) compact, and \( K \subset W, \overline{W} \subset X \) open. Then there is an open \( V \supset K \) with \( \overline{V} \subset W, \overline{V} \) compact, and an \( f : X \to [0, 1] \) with \( f \equiv 1 \) in a neighborhood of \( K \) and \( \text{spt } f \subset V \).

**Proof:** By Lemma 2.3 each \( x \in K \) has a neighborhood \( U_x \) with \( \overline{U}_x \subset W \) and \( \overline{U}_x \) compact. Then by compactness of \( K \) we have \( K \subset V = \bigcup_{j=1}^N U_{x_j} \) for some finite collection \( x_1, \ldots, x_N \in K \) and \( \overline{V} = \bigcup_{j=1}^N \overline{U}_{x_j} \subset W \). Now \( \overline{V} \) is compact, so by Corollary 2.2 it is a normal space and the Urysohn lemma can be applied to give \( f_0 : \overline{V} \to [0, 1] \) with \( f_0 \equiv 1 \) on \( K \) and and \( f_0 \equiv 0 \) on \( \overline{V} \setminus V \). Then of course the function \( f_1 \) defined by \( f_1 \equiv f_0 \) on \( \overline{V} \) and \( f_1 \equiv 0 \) on \( X \setminus \overline{V} \) is continuous (check!) because \( f \mid \overline{V} \) is continuous and \( f \) is identically zero (the value of \( f \mid X \setminus \overline{V} \) on the overlap set \( \overline{V} \setminus V \equiv \overline{V} \cap (X \setminus V) \)). Finally we let \( f = 2 \min\{f_1, \frac{1}{2}\} \) and observe that \( f \) is then identically 1 in the set where \( f_1 > \frac{1}{2} \), which is an open set containing \( K \), and \( f \) evidently has all the remaining stated properties.

The following corollary of Lemma 2.4 is important:

### 2.5 Corollary (Partition of Unity)

If \( X \) is a locally compact Hausdorff space, \( K \subset X \) is compact, and if \( U_1, \ldots, U_N \) is any open cover for \( K \), then there exist continuous \( \psi_j : X \to [0, 1] \) such that \( \text{spt } \psi_j \) is a compact subset of \( U_j \) for each \( j \), and \( \sum_{j=1}^N \psi_j \equiv 1 \) in a neighborhood of \( K \).

**Proof:** By Lemma 2.3, for each \( x \in K \) there is a \( j \in \{1, \ldots, N\} \) and a neighborhood \( U_x \) of \( x \) such that \( \overline{U}_x \) is a compact subset of this \( U_j \). By compactness of \( K \) we have finitely many of these neighborhoods, say \( U_{x_1}, \ldots, U_{x_N} \), with \( K \subset \bigcup_{j=1}^N U_{x_j} \). Then for each \( j = 1, \ldots, N \) we define \( V_j \) to be the union of all \( U_{x_j} \) such that \( \overline{U}_{x_j} \subset U_j \). Then the \( \overline{V}_j \) is a compact subset of \( U_j \) for each \( j \), and the \( V_j \) cover \( K \). So by Lemma 2.4 for each \( j = 1, \ldots, N \) we can select \( \psi_j : X \to [0, 1] \) with \( \psi_j \equiv 1 \) on \( \overline{V}_j \) and \( \psi_j \equiv 0 \) on \( X \setminus W_j \) for some open \( W_j \) with \( \overline{W}_j \) a compact subset of \( U_j \) and \( W_j \supset V_j \). We can also use Lemma 2.4 to select \( \phi_0 : X \to [0, 1] \) with \( \phi_0 \equiv 0 \) in a neighborhood of \( K \) and \( \phi_0 \equiv 1 \) outside a compact subset of \( \bigcup_{j=1}^N V_j \). Then by construction \( \sum_{j=0}^N \psi_j > 0 \) everywhere on \( X \), so we can define continuous functions \( \varphi_j \) by

\[
\varphi_j = \frac{\psi_j}{\sum_{i=0}^N \psi_i}, \quad j = 1, \ldots, N.
\]

Evidently these functions have the required properties.

We now give the definition of Radon measure. Radon measures are typically used only in locally compact Hausdorff space, but the definition and the first two lemmas following it are valid in arbitrary Hausdorff space:

### 2.6 Definition

Given a Hausdorff space \( X \), a “Radon measure” on \( X \) is an outer measure \( \mu \) on \( X \) having the 3 properties:

\[
\mu \text{ is Borel regular and } \mu(K) < \infty \text{ for compact } K \subset X \quad (R1)
\]

\[
\mu(A) = \lim_{U \text{ open, } U \supseteq A} \mu(U) \text{ for each subset } A \subset X \quad (R2)
\]

\[
\mu(U) = \sup_{K \text{ compact, } K \subset U} \mu(K) \text{ for each open } U \subset X. \quad (R3)
\]

Such measures automatically have a property like (R3) with an arbitrary \( \mu \)-measurable subset of finite measure.
2.7 Lemma. Let $X$ be a Hausdorff space and $\mu$ a Radon measure on $X$. Then $\mu$ automatically has the property

$$\mu(A) = \sup_{K \subset A, K \text{ compact}} \mu(K)$$

for every $\mu$-measurable set $A \subset X$ with $\mu(A) < \infty$.

Proof: Let $\varepsilon > 0$. By definition of Radon measure we can choose an open $U$ containing $A$ with $\mu(U \setminus A) < \varepsilon$, and then a compact $K \subset U$ with $\mu(U \setminus K) < \varepsilon$ and finally an open $W$ containing $U \setminus A$ with $\mu(W \setminus (U \setminus A)) < \varepsilon$ (so that $\mu(W) \leq \varepsilon + \mu(U \setminus A) < 2\varepsilon$). Then we have that $K \setminus W$ is a compact subset of $U \setminus W$, which is a subset of $A$, and also

$$\mu(A \setminus (K \setminus W)) \leq \mu(U \setminus (K \setminus W)) \leq \mu(U \setminus K) + \mu(W) \leq 3\varepsilon,$$

which completes the proof. □

The following lemma asserts that the defining property (R1) of Radon measures follows automatically from the remaining two properties ((R2) and (R3)) in case $\mu$ is finite and additive on finite disjoint unions of compact sets.

2.8 Lemma. Let $X$ be a Hausdorff space and assume that $\mu$ is an outer measure on $X$ satisfying the properties (R2), (R3) above, and in addition assume that

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) < \infty, \ K_1, K_2 \text{ compact}, \ K_1 \cap K_2 = \emptyset.$$

Then $\mu$ is Borel regular, hence (R1) holds, hence $\mu$ is a Radon measure.

Proof: Note that (R2) implies that for every set $A \subset X$ we can find open sets $U_j$ such that $A \subset \cap_j U_j$ and $\mu(A) = \mu(\cap_j U_j)$. So to complete the proof of (R1) we just have to check that all Borel sets are $\mu$-measurable; since the $\mu$-measurable sets form a $\sigma$-algebra and the Borel sets form the smallest $\sigma$-algebra which contains all the open sets, we thus need only to check that all open sets are $\mu$-measurable.

Let $\varepsilon > 0$ be arbitrary, $Y$ an arbitrary subset of $X$ with $\mu(Y) < \infty$ and let $U$ be an arbitrary open subset of $X$. By (R2) we can pick an open set $V \supset Y$ with $\mu(V) < \mu(Y) + \varepsilon$ and by (R3) we can pick a compact set $K_1 \subset V \cap U$ with $\mu(V \cap U) \leq \mu(K_1) + \varepsilon$, and then a compact set $K_2 \subset V \setminus K_1$ with $\mu(V \setminus K_2) \leq \mu(K_2) + \varepsilon$. Then

$$\mu(V \setminus K_1) \leq \mu(K_2) + \varepsilon.$$

Choosing $\varepsilon$ such that $\mu(V \setminus K_1) < \mu(Y)$, we get

$$\mu(V \setminus K_1) < \mu(Y) \Rightarrow \mu(V \setminus K_1) < \mu(K_2) + \varepsilon.$$
and \( \mu(V_j) < \infty \) for each \( j \), and so in this case (when \( X \) is \( \sigma \)-compact) the identity in (1) holds for every subset \( A \subset X \); that is

\[
\mu(A) = \inf_{U \text{ open}, A \subset U} \mu(U) \text{ for every } A \subset X,
\]

which is the property \((R2)\). Next we note that if \( A \subset X \) is \( \mu \)-measurable, then we can write \( A = \bigcup_j A_j \), where \( A_j = A \cap K_j \) (because \( X = \bigcup_j K_j \) and \( \mu(A_j) \leq \mu(K_j) < \infty \) for each \( j \), so (2) actually holds for every \( \mu \)-measurable \( A \) in case \( X \) is \( \sigma \)-compact (i.e. in case \( X = \bigcup_{i=1}^\infty K_i \) with \( K_i \) compact), and for any closed set \( C \) we can write \( C = \bigcup_j C_j \) where \( C_j \) is the increasing sequence of compact sets given by \( C_j = C \cap (\bigcup_{i=1}^j K_i) \) and so \( \mu(C) = \lim_j \mu(C_j) \) and hence \( \mu(C) = \sup_{K \subset C, K \text{ compact}} \mu(K) \). Thus in the \( \sigma \)-compact case (2) actually tells us that \( \mu(A) = \sup_{K \subset A, K \text{ compact}} \mu(K) \) for any \( \mu \)-measurable set \( A \). This in particular holds for \( A = \text{ an open set}, \) which is the remaining property \((R3)\) we needed. \( \square \)

The following result is one of the main theorems related to Radon measures, asserting that for a Radon measure \( \mu \) on a locally compact Hausdorff space, the continuous functions with compact support are dense in \( L^p(\mu), 1 \leq p < \infty \).

Here and subsequently we use the notation

\[
C_c(X) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous with spt}\ f \text{ compact} \},
\]

where \( \text{spt} f \text{ is support of } f = \text{ closure of } \{ x \in X \mid f(x) \neq 0 \} \).

2.10 Theorem. Let \( X \) be a locally compact Hausdorff space, \( \mu \) a Radon measure on \( X \) and \( 1 \leq p < \infty \). Then \( C_c(X) \) is dense in \( L^p(\mu) \); that is, for each \( \varepsilon > 0 \) and each \( f \in L^p \) there is a \( g \in C_c(X) \) such that \( \| g - f \|_p < \varepsilon \).

In view of Lemma 2.9 and the fact that to every Borel measure \( \mu \) on a topological space \( X \) (i.e. every map \( \mu : \{ \text{all Borel sets of } X \} \to [0, \infty] \) with \( \mu(\emptyset) = 0 \) and \( \mu(\bigcup_{j=1}^\infty B_j) = \sum_{j=1}^\infty \mu(B_j) \) for every pairwise disjoint collection of Borel sets \( B_1, B_2, \ldots \)), there is a Borel regular outer measure \( \overline{\mu} \) on \( X \) defined by \( \overline{\mu}(A) = \inf_{B \supset A : B \text{ Borel}} \mu(B) \), we see that Theorem 2.10 directly implies the following important corollary:

2.11 Corollary. If \( X \) is a locally compact Hausdorff space such that every open set in \( X \) is the countable union of compact sets, and if \( \mu \) is any Borel measure on \( X \) which is finite on each compact set, then the space \( C_c(X) \) is dense in \( L^1(\mu) \) and \( \mu \) is the restriction to the Borel sets of a Radon measure \( \overline{\mu} \).

Proof of Theorem 2.10: Let \( f : X \to \mathbb{R} \) be \( \mu \)-measurable with \( \| f \|_p < \infty \) and let \( \varepsilon > 0 \). Observe that the simple functions are dense in \( L^p(\mu) \) (which one can check using the dominated convergence theorem and the fact that both \( \int f_+ \) and \( \int f_- \) can be expressed as the pointwise limits of increasing sequences of non-negative simple functions), so we can pick a simple function \( \varphi = \sum_{j=1}^J a_j \chi_{A_j}, \) where the \( a_j \) are distinct non-zero reals and \( A_j \) are pairwise disjoint \( \mu \)-measurable subsets of \( X \), such that \( \| f - \varphi \|_p < \varepsilon \). Since \( \| \varphi \|_p \leq \| f - \varphi \|_p + \| f \|_p < \infty \) we must then have \( \mu(A_j) < \infty \) for each \( j \).

Pick \( M > \max\{|a_1|, \ldots, |a_J|\} \) and use Lemma 2.7 to select compact \( K_j \subset A_j \) with \( \mu(A_j \setminus K_j) < \varepsilon p/(2p+1)M^pN^p \). Also, using the definition of Radon measure, we can find open \( U_j \supset K_j \) with \( \mu(U_j \setminus K_j) < \varepsilon p/(2p+1)M^pN^p \) and by Lemma 2.7 we can assume without loss of generality that these open sets \( U_1, \ldots, U_N \) are pairwise disjoint (otherwise replace \( U_j \) by \( U_j \cap U_0^c \), where \( U_0^c \) are pairwise disjoint open sets with \( \mu(U_1^c, \ldots, U_N^c) \) by \( K_j \subset U_0^c \)). By Lemma 2.4 we have \( g_j \in C_c(X) \) with \( g_j \equiv a_j \) on \( K_j \), \( \{ x : g_j(x) \neq 0 \} \) contained in a compact subset of \( U_j \), and \( \sup |g_j| \leq |a_j| \), and hence by the pairwise disjointness of the \( U_j \) we have that \( g \equiv \sum_{j=1}^J g_j \equiv \varphi \) on each \( K_j \) and \( \sup |g| \equiv \sup |\varphi| < M \). Then \( \varphi - g \) vanishes off the set \( \bigcup_j (U_j \setminus K_j) \cup (A_j \setminus K_j) \) and we have \( \int_X |\varphi - g|^p d\mu \leq \sum_j \int_{U_j \setminus K_j \cup (A_j \setminus K_j)} |\varphi - g|^p d\mu \leq (2M)^p \sum_j (\mu(A_j \setminus K_j) + \mu(U_j \setminus K_j)) \leq 2\varepsilon, \) and hence \( \| f - g \|_p \leq \| f - \varphi \|_p + \| \varphi - g \|_p \leq 2\varepsilon, \) as required.

We now state the Riesz representation theorem for non-negative functionals on the space \( K_+ \), where, here and subsequently, \( K_+ \) denotes the set of non-negative \( C_c(X, \mathbb{R}) \) functions, i.e. the set of continuous functions \( f : X \to [0, \infty) \) with compact support.

2.12 Theorem (Riesz for non-negative functionals.) Suppose \( X \) is a locally compact Hausdorff space, \( \lambda : K_+ \to [0, \infty) \) with \( \lambda(cf) = c\lambda(f) \), \( \lambda(f + g) = \lambda(f) + \lambda(g) \) whenever \( c \geq 0 \) and \( f, g \in K_+ \), where \( K_+ \) is the set of all non-negative continuous functions \( f \) on \( X \) with compact support. Then there is a Radon measure \( \mu \) on \( X \) such that \( \lambda(f) = \int_X f d\mu \) for all \( f \in K_+ \).

Before we begin the proof of 2.12 we the following preliminary observation:

2.13 Remark: Observe that if \( f, g \in K_+ \) with \( f \leq g \) then \( g - f \in K_+ \) and hence \( \lambda(g) = \lambda(f + (g - f)) = \lambda(f) + \lambda(g - f) \geq \lambda(f) \), so

\[
(*) \quad f, g \in K_+ \text{ with } g \equiv 1 \text{ on spt } f \Rightarrow \\
\lambda(f) \leq (\text{sup } f) \lambda(g), \quad f \in K_+, \text{ spt } f \subset K.
\]

because \( f g \equiv f \) and \( f \leq (\text{sup } f) g \).

Proof of Theorem 2.12: For \( U \subset X \) open, we define

\[
(1) \quad \mu(U) = \sup_{f \in K_+, f \leq 1, \text{ spt } f \subset U} \lambda(f),
\]

and the following theorems: 2.10, 2.11, and 2.12, which are essential tools for the study of Radon measures and their applications in measure theory and functional analysis.
and for arbitrary \( A \subseteq X \) we define
\[
\mu(A) = \lim_{U \supset A, U \text{ open}} \mu(U).
\]

Notice that these definitions are consistent when \( A \) is itself open, and of course the definitions (1),(2) guarantee \( \mu(\emptyset) = 0 \) and that \( \mu \) is monotone—i.e.

(3) \( A \subseteq B \Rightarrow \mu(A) \leq \mu(B). \)

Also if \( f \in \mathcal{K}_+ \) with \( f \leq 1 \) and \( V \) is open with \( V \supset f \) then by (1) \( \lambda(f) \leq \mu(V) \), and hence, taking inf over such \( V \) and using (2), we see
\[
f \in \mathcal{K}_+ \text{ with } f \leq 1 \Rightarrow \lambda(f) \leq \mu(\text{spt } f).
\]

and then for any open \( U \) we can use (1) and (4) to conclude
\[
\mu(U) = \sup_{f \in \mathcal{K}_+, \text{spt } f \subseteq U} \mu(\text{spt } f).
\]

Notice next that if \( K \) is compact then, by Lemma 2.4, if \( W \supset K \) is open there is \( g \in \mathcal{K}_+ \) with \( g \equiv 1 \) in a neighborhood \( V \) of \( K \) and with \( g \leq 1 \) and \( \text{spt } g \Subset W \). Then by (3),(1) and (**) we have, for any such \( g \),
\[
\mu(K) \leq \mu(V) = \sup_{f \in \mathcal{K}_+, \text{spt } f \subseteq V} \lambda(f) \leq \lambda(g) \leq \mu(W).
\]

To prove that \( \mu \) is an outer measure it still remains to check countable subadditivity. To see this, first let \( U_1, U_2, \ldots \) be open and \( U = \bigcup U_j \), then for any \( f \in \mathcal{K}_+ \) with \( \sup f \leq 1 \) and \( \text{spt } f \subseteq U \) we have, by compactness of \( \text{spt } f \), that \( \text{spt } f \subseteq \bigcup_{j=1}^N U_j \) for some integer \( N \), and by using a partition of unity \( \varphi_1, \ldots, \varphi_N \) for \( \text{spt } f \) subordinate to \( U_1, \ldots, U_N \) (see Corollary 2.5), we have
\[
\lambda(f) = \sum_{j=1}^N \lambda(\varphi_j f) = \sum_{j=1}^N \mu(U_j).
\]

Taking sup over all such \( f \) we then have
\[
\mu(U) \leq \sum_{j=1}^N \mu(U_j).
\]

It then easily follows by applying definitions (1),(2) that \( \mu(\bigcup A_j) \leq \sum \mu(A_j) \). So indeed \( \mu \) is an outer measure on \( X \).

Finally we want to show that \( \mu \) is a Radon measure. For this we are going to use Lemma 2.8 above, so we have to check the hypotheses of Lemma 2.8. Hypothesis (R2) needed for Lemma 2.8 is true by definition and (R3) is true by (5). Since we also have finiteness of \( \mu(K) \) for compact \( K \) by (6), it remains only to prove the additivity property

(7) \( K_1, K_2 \text{ disjoint compact sets in } X \Rightarrow \mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2). \)

To check this, let \( U \) be any open set containing \( K_1 \cup K_2 \) and use Corollary 2.2 to choose disjoint open \( V_j \supset K_j \) with \( V_j \subseteq U \), \( j = 1, 2 \). Then by (3) and (1)
\[
\mu(K_1) + \mu(K_2) \leq \mu(V_1) + \mu(V_2) = \sup_{g_j \in \mathcal{K}_+, \text{spt } g_j \subseteq V_j, g_j \leq 1, j = 1, 2} \lambda(g_1) + \lambda(g_2)
\]
\[
= \sup_{g_j \in \mathcal{K}_+, \text{spt } g_j \subseteq V_j, g_j \leq 1, j = 1, 2} \lambda(g_1 + g_2)
\]

On the other hand \( g_1 + g_2 \leq 1 \) on \( X \) (because \( V_1 \cap V_2 = \emptyset \)) and so
\[
\sup_{g_j \in \mathcal{K}_+, \text{spt } g_j \subseteq V_j, g_j \leq 1, j = 1, 2} \lambda(g_1 + g_2) \leq \sup_{f \in \mathcal{K}_+, \text{spt } f \subseteq U, f \leq 1} \lambda(f) = \mu(U).
\]

Hence we have proved that \( \mu(K_1) + \mu(K_2) \leq \mu(U) \), and taking inf over all open \( U \supset K_1 \cup K_2 \) we have by (2) that \( \mu(K_1) + \mu(K_2) \leq \mu(K_1 \cup K_2) \), and of course the reverse inequality holds by subadditivity of \( \mu \), hence the hypotheses of Lemma 2.8 are all established and \( \mu \) is a Radon measure.

Next observe that by (4) we have \( \lambda(h) = \lim_{n \to \infty} \lambda(\max\{h - 1/n, 0\}) \leq \mu(\{x : h(x) > 0\}) \), and hence
\[
\lambda(h) = \lim_{n \to \infty} \lambda(\max\{h - 1/n, 0\}) \leq \mu(\{x : h(x) > 0\}) \sup h, \ h \in \mathcal{K}_+.
\]

since \( h \) is the uniform limit of \( \max\{h - 1/n, 0\} \) in \( X \) and \( \sup \max\{h - 1/n, 0\} \subseteq \{x : h(x) > 0\} \) for each \( n \). For \( f \in \mathcal{K}_+ \) (not identically zero) and \( \varepsilon > 0 \), we let \( M = \sup f \) can select points \( 0 = t_0 < t_1 < t_2 < \ldots < t_k < \varepsilon \) for each \( j = 1, \ldots, N \) and with \( \mu(\{f^{-1}\{t_j\}\}) = 0 \) for each \( j = 1, \ldots, N \). Notice that the latter requirement is no problem because \( \mu(\{f^{-1}\{t_j\}\}) = 0 \) for all but a countable set of \( t > 0 \), by virtue of the fact that \( \mu(\{x \in X : f(x) > 0\}) \leq \mu(\text{spt } f) < \infty \).

Now let \( U_j = f^{-1}\{\{j-1\}, t_j\}, j = 1, \ldots, N \). (Notice that then the \( U_j \) are pairwise disjoint and each \( U_j \subseteq K \), where \( K \), compact, is the support of \( f \).

Now by the definition (1) we can find \( g_j \in \mathcal{K}_+ \) such that \( g_j \leq 1 \), \( \text{spt } g_j \subseteq U_j \), and \( \lambda(g_j) \geq \mu(U_j) - \varepsilon/N \). Also for any compact \( K_j \subseteq U_j \) we can construct a function \( h_j \in \mathcal{K}_+ \) with \( h_j \equiv 1 \) in a neighborhood of \( K_j \cup \text{spt } g_j \), \( \text{spt } h_j \subseteq U_j \), and \( h_j \leq 1 \) everywhere. Then \( h_j \geq g_j, h_j \leq 1 \) everywhere and \( \text{spt } h_j \) is a compact subset of \( U_j \) and so
\[
\mu(U_j) - \varepsilon/N \leq \lambda(g_j) \leq \lambda(h_j) \leq \mu(U_j), \ j = 1, \ldots, N.
\]

Since \( \mu \) is a Radon measure, we can in fact choose the compact \( K_j \subseteq U_j \) such that \( \mu(U_j \setminus K_j) < \varepsilon/N \). Then, because \( \{x : (f - \sum_{j=1}^N h_j)(x) > 0\} \subset \bigcup_{j=1}^N (U_j \setminus K_j) \) by (8) we have
\[
\lambda(f - \sum_{j=1}^N h_j) \leq M \sum_{j=1}^N \mu(U_j \setminus K_j) \leq \varepsilon M.
\]

Then by using (9), (10) and the linearity of \( \lambda \) (together with the fact \( t_{j-1}h_j \leq f h_j \leq t_jh_j \) for each \( j = 1, \ldots, N \)), we see that
\[
\sum_{j=1}^N t_j^{-1} \mu(U_j) - M \sum_{j=1}^N \mu(U_j \setminus K_j) h_j \leq \lambda(f - \sum_{j=1}^N h_j) + \varepsilon M \leq \sum_{j=1}^N t_j^{-1} \mu(U_j) + \varepsilon M.
\]

Since trivially
\[
\sum_{j=1}^N t_j^{-1} \mu(U_j) \leq f_x f d \mu \leq \sum_{j=1}^N t_j^{-1} \mu(U_j),
\]
we then have
\[ -\varepsilon (\mu(K) + M) \leq -\sum_{j=1}^{N} (1 \cdot t_j - t_{j-1}) \mu(U_j) - \varepsilon M \]
\[ \leq \int_X f \, d\mu - \lambda(f) \]
\[ \leq \sum_{j=1}^{N} (1 \cdot t_j - t_{j-1}) \mu(U_j) + \varepsilon M \]
\[ \leq \varepsilon (\mu(K) + M), \]

where \( K = \text{spt} \, f \). This completes the proof of 2.12. \( \Box \)

We can now state the Riesz Representation Theorem. In the statement, \( C_c(X, \mathbb{R}^n) \) will denote the set of vector functions \( f : X \to \mathbb{R}^n \) which are continuous and which have compact support. (That is \( f \equiv 0 \) outside a compact subset of \( X \).)

2.14 Theorem (Riesz Representation Theorem.) Suppose \( X \) is a locally compact Hausdorff space, and \( L : C_c(X, \mathbb{R}^n) \to \mathbb{R} \) is linear with
\[ \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq 1} L(f) < \infty \text{ whenever } K \subset X \text{ is compact.} \]
Then there is a Radon measure \( \mu \) on \( X \) such that for each compact \( K \subset X \) there is a vector function \( v : X \to \mathbb{R}^n \) with \( |v| = 1 \) everywhere and \( v_j \mu \)-measurable, \( j = 1, \ldots, n \), and with
\[ L(f) = \int_X f \cdot v \, d\mu \text{ for any } f \in C_c(X, \mathbb{R}^n) \text{ with } \text{spt} \, f \subset K. \]

In the cases when \( X \) is \( \sigma \)-compact (i.e. \( \exists \) compact \( K_1, K_2, \ldots \) with \( X = \bigcup_j K_j \)) or \( L \) is bounded (i.e. \( \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq 1} |L(f)| < \infty \)), \( v \) can be chosen independent of \( K \).

Proof: We first define
\[ \lambda(h) = \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq h} L(f) \]
for any \( h \in \mathcal{K}_+ \). We claim that \( \lambda \) has the linearity properties of Lemma 2.12. Indeed it is clear that \( \lambda(ch) = c \lambda(h) \) for any constant \( c \geq 0 \) and any \( h \in \mathcal{K}_+ \). Now let \( g, h \in \mathcal{K}_+ \), and notice that if \( f_1, f_2 \in C_c(X, \mathbb{R}^n) \) with \( |f_1| \leq g \) and \( |f_2| \leq h \), then \( |f_1 + f_2| \leq g + h \) and hence \( \lambda(g + h) \geq L(f_1) + L(f_2) \). Taking sup over all such \( f_1, f_2 \) we then have \( \lambda(g + h) \geq \lambda(g) + \lambda(h) \). To prove the reverse inequality we let \( f \in C_c(X, \mathbb{R}^n) \) with \( |f| \leq g + h \), and define
\[ f_1 = \begin{cases} \frac{g}{g+h} f & \text{if } g+h > 0, \\ 0 & \text{if } g+h = 0, \end{cases} \quad f_2 = \begin{cases} \frac{h}{g+h} f & \text{if } g+h > 0, \\ 0 & \text{if } g+h = 0. \end{cases} \]

Then \( f_1 + f_2 = f, \ |f_1| \leq g, \ |f_2| \leq h \) and it is readily checked that \( f_1, f_2 \in C_c(X, \mathbb{R}^n) \). Then \( L(f) = L(f_1) + L(f_2) \leq \lambda(g) + \lambda(h) \), and hence taking sup over all such \( f \) we have \( \lambda(g + h) \leq \lambda(g) + \lambda(h) \). Therefore we have \( \lambda(g + h) = \lambda(g) + \lambda(h) \) as claimed. Thus \( \lambda \) satisfies the conditions of the Theorem 2.12, hence there is a Radon measure \( \mu \) on \( X \) such that
\[ \lambda(h) = \int_X h \, d\mu, \quad h \in \mathcal{K}_+, \quad j = 1, \ldots, n. \]

That is, we have
\[ (\dagger) \sup \left\{ L(f) = \int_X h \, d\mu, \quad h \in \mathcal{K}_+ \right\} \]
Thus if \( j \in \{1, \ldots, n\} \) we have in particular (since \( |fe_j| = |f| \in \mathcal{K}_+ \) for any \( f \in C_c(X, \mathbb{R}^n) \))
\[ |L(fe_j)| \leq \int_X |f| \, d\mu \equiv \|f\|_{L^1(\mu)} \quad \forall f \in C_c(X, \mathbb{R}). \]
Thus \( \int_X \) extends to a bounded linear functional on \( L^1(\mu) \). In either of the \( \text{3 cases (i) } K \text{ compact is given and we use Riesz Representation Theorem for } L^1(\mu \subset K), \text{ or (ii) } \|\|L\|\| = \mu(X) < \infty \) and we use Riesz Representation Theorem for the finite measure case, or (iii) \( X = \bigcup_{j=1}^\infty K_j \) with \( K_j \) compact for each \( j \) and we use Riesz Representation Theorem for the \( \sigma \)-finite case, we know that there is a bounded \( \mu \)-measurable function \( v \) such that
\[ L(fe_j) = \int_X f(v) \, d\mu, \quad f \in C_c(X, \mathbb{R}), \]
where in case (i) we impose the additional restriction \( \text{spt} \, f \subset K \). Since any \( f = (f_1, \ldots, f_n) \) can be expressed as \( f = \sum^n_{j=1} f_j e_j \), we thus deduce
\[ (*) \quad L(f) = \int_X f \cdot v \, d\mu, \quad f \in C_c(X, \mathbb{R}^n), \]
where \( v = (v_1, \ldots, v_n) \), and so by (\dagger)
\[ \int_X h \, d\mu = \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq h} \int_X f \cdot v \, d\mu = \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq 1, g \in \mathcal{K}_+, g \leq h} \int_X g f \cdot v \, d\mu \]
for every \( h \in \mathcal{K}_+ \), where in case (i) we assume \( \text{spt} \, h \subset K \). Now \( |f \cdot v| \leq |f| |v| \) so we have
\[ \sup_{f \in C_c(X, \mathbb{R}^n), |f| \leq 1, g \in \mathcal{K}_+, g \leq h} \int_X g f \cdot v \, d\mu \leq \sup_{g \in \mathcal{K}_+, g \leq h} \int_X g |v| \, d\mu = \int_X h|v| \, d\mu \]
Since \( C_c(X) \) is dense in \( L^1(\mu) \), we can choose a sequence \( f_k \) with \( |f_k| = 1 \) and \( f_k \cdot v \to |v| \) on \( \text{spt} \, h \), so the bound on the right of the previous inequality is attained and we have proved
\[ \int_X h \, d\mu = \int_X h|v| \, d\mu \]
and again using the density of \( C_c(X) \) in \( L^1(\mu) \) we have \( |v| = 1 \mu \text{-a.e.} \) \( \Box \)
We conclude with an important compactness theorem for Radon Measures.

Recall that Alaoglu's theorem (see e.g. Royden, “Real Analysis” 3rd Edition, Macmillan 1988, p.237), which is a corollary of Tychonoff’s theorem, tells us that the closed unit ball in the dual space of a normed linear space must be weak* compact: that is, given any normed linear space $X$ with dual space $X^*$ (i.e. $X^*$ is the normed space consisting of all the bounded linear functionals $F : X \to \mathbb{R}$), then $\{ F \in X^* : \| F \| \leq 1 \}$ is weak* compact, meaning that for any sequence $F_j \in X^*$ with $\sup_j \| F_j \| < \infty$ there is a subsequence $F_{j_k}$ and an $F \in X^*$ with $F_{j_k}(x) \to F(x)$ for each fixed $x \in X$.

In particular if $X$ is compact and $X = C(X)$ (the continuous real-valued functions on $X$) $\{ \lambda \in X^* : \| \lambda \| \leq 1 \}$ is weak* compact. That is, given a sequence $\{ \lambda_k \}$ of bounded linear functionals on $C(X)$ with $\sup_{k \geq 1} \| \lambda_k \| < \infty$, we can find a subsequence $\{ \lambda_{k'} \}$ and bounded linear functional $\lambda$ such that $\lambda_{k'}(f) = \lambda(f)$ for each fixed $f \in C(X)$. Using the above Riesz Representation 2.12, this implies the following assertion concerning sequences of Radon measures on $X$, assuming $X$ is $\sigma$-compact.

**2.15 Theorem (Compactness Theorem for Radon Measures.)** Suppose $\{ \mu_k \}$ is a sequence of Radon measures on the locally compact, $\sigma$-compact Hausdorff space $X$ with the property $\sup_k \mu_k(K) < \infty$ for each compact $K$. Then there is a subsequence $\{ \mu_{k'} \}$ which converges to a Radon measure $\mu$ on $X$ in the sense that

$$\lim_{k'} \int_X f \, d\mu_{k'} = \int_X f \, d\mu, \quad \text{for each } f \in C_c(X).$$

**Proof:** Let $K_1, K_2, \ldots$ be an increasing sequence of compact sets with $X = \bigcup_j K_j$ and let $F_{j,k} : C(K_j) \to \mathbb{R}$ be defined by $F_{j,k}(f) = \int_{K_j} f \, d\mu_k$, $k = 1, 2, \ldots$. By the Alaoglu theorem there is a subsequence $F_{j,k'}$ and a non-negative bounded functional $F_j : C(K_j) \to \mathbb{R}$ with $F_{j,k'}(f) \to F_j(f)$ for each $f \in C(K_j)$. By choosing the subsequences successively and taking a diagonal sequence we then get a subsequence $\mu_{k'}$ and a non-negative linear $F : C_c(X) \to \mathbb{R}$ with $\int_X f \, d\mu_{k'} \to F(f)$ for each $f \in C_c(X)$, where $F(f) = F_j(f|K_j)$ whenever $\text{spt } f \subset K_j$. (Notice that this is unambiguous because if $\text{spt } f \subset K_j$ and $\ell > j$ then $F_{\ell}(f|K_j) = F_j(f|K_j)$ by construction.) Then by applying Theorem 2.12 we have a Radon measure $\mu$ on $X$ such that $F(f) = \int_X f \, d\mu$ for each $f \in C_c(X)$, and so $\int_X f \, d\mu_{k'} \to \int_X f \, d\mu$ for each $f \in C_c(X)$. 