MATH 215B. SOLUTIONS TO HOMEWORK 1

1. (6 marks) Show that a space X is contractible iff every map $f : X \to Y$, for arbitrary Y, is nullhomotopic. Similarly, show X is contractible iff every map $f : Y \to X$ is nullhomotopic.

Solution

Suppose every map $f : X \to Y$, for all Y, is nullhomotopic. Then in particular the identity $X \to X$ is nullhomotopic, and so X is contractible.

Similarly, suppose every map $f : Y \to X$, for all Y, is nullhomotopic. Then in particular the identity $X \to X$ is nullhomotopic, and so X is contractible.

Now suppose X is contractible. I.e., there is a point $a \in X$ and a homotopy $h : X \times I \to X$ such that h_0 is the identity on X and h_1 is the constant map with value a.

If, furthermore, $f : X \to Y$, then $f \circ h : X \times I \to Y$ is a homotopy from f to the constant map with value $f(a)$. Thus f is nullhomotopic.

If, on the other hand, $f : Y \to X$, then the map

$$\begin{aligned} Y \times I &\to X \\ (y, t) &\mapsto h(f(y), t) \end{aligned}$$

is a homotopy from f to the constant map with value a. Thus f is nullhomotopic.

2. (8 marks) Show that the cardinality of the set of path components is a homotopy invariant.

Solution

For X a space, let $\pi_0(X)$ denote the set of path components of X. In this problem, for $x \in X$, let $[x]$ denote the path-component of X containing x.

For $f : X \to Y$, there is an induced map $f_* : \pi_0(X) \to \pi_0(Y)$ of *sets* defined by $f_*([x]) = [f(x)]$.

In order to show that this is well-defined, we must show that if points x and x' belong to the same path-component of X, then $f(x)$ and $f(x')$ belong to the same path-component of Y. Indeed, if x and x' are connected by a path $\phi : I \to X$, then $f \circ \phi$ is a path from $f(x)$ to $f(x')$.

Furthermore, note that if $g : Y \to Z$, then $(g \circ f)_* = g_* \circ f_*$.

Furthermore, if $h : X \times I \to Y$ is a homotopy, then $(h_0)_* = (h_1)_*$. To see this, note that for $x \in X$, we have a path

$$\begin{aligned} I &\to Y \\ t &\mapsto h(x, t) \end{aligned}$$

from $h_0(x)$ to $h_1(x)$. So $[h_0(x)] = [h_1(x)]$, and so $(h_0)_*([x]) = (h_1)_*([x])$.

Finally, if $f : X \to Y$ is a homotopy equivalence with homotopy inverse g, then the homotopies $fg \simeq \mathbb{1}_Y$, $gf \simeq \mathbb{1}_X$ induce equalities $f_* g_* = \mathbb{1}_{\pi_0(Y)}$, $g_* f_* = \mathbb{1}_{\pi_0(X)}$. Thus f_* is a bijection between the sets of path components.

3. (12 marks) Show that for a space X, the following three conditions are equivalent:

(a) Every map $S^1 \to X$ is homotopic to a constant map, with image a point.
(b) Every map $S^1 \to X$ extends to a map $D^2 \to X$.
(c) $\pi_1(X, x_0) = 0$ for all $x_0 \in X$.

Deduce that a space X is simply-connected iff all maps $S^1 \to X$ are homotopic. [In this problem, "homotopic" means "homotopic without regard to basepoints".] [Fun fact: This exercise essentially shows that a space is simply-connected if and only if it has no "holes".]

Solution

(a) $\Rightarrow$ (b): Suppose $f : S^1 \to X$. By hypothesis, there's a homotopy $h : S^1 \times I \to X$ from f to a constant map. That is, $h_0 = f$ and there is a point $x \in X$ such that, for all $s \in S^1$, $h(s, 1) = x$. Because of the latter condition, h factors through the quotient $S^1 \times I/S^1 \times \{1\}$. That is, h is equal to a composition

$$S^1 \times I \to S^1 \times I/S^1 \times \{1\} \to X$$

where the first map is the quotient map.

The pair $(S^1 \times I/S^1 \times \{1\}, S^1 \times \{0\})$ is homeomorphic to (D^2, S^1). The homeomorphism is induced by the map $\Phi : S^1 \times I \to D^2$ defined by $\Phi(a, x) = a \cdot (1 - x)$, where we look at a and x as complex numbers. So the second map above gives a map $D^2 \to X$ such that the restriction to S^1 is equal to f.

(b) $\Rightarrow$ (c): As in the next problem, we can regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. Suppose $f : (S^1, s_0) \to (X, x_0)$ is such a map. By hypothesis, it extends to a map $h : (D^2, s_0) \to (X, x_0)$. Let ϕ be a deformation-retraction of D^2 onto s_0. Then

$$\begin{aligned} S^1 \times I &\to X \\ (s, t) &\mapsto f(\phi_t(s)) \end{aligned}$$

is a basepoint-preserving homotopy of f to the constant map with value x_0. So all loops in X are nullhomotopic, and so $\pi_1(X, x_0)$ is trivial.

(Here is an easy way to get a deformation-retraction of D^2 onto a point s_0 in its boundary: Identify D^2 with the disk of unit radius in the plane with center at $(1, 0)$. Then

$$\begin{aligned} D^2 \times I &\to D^2 \\ (\vec{d}, t) &\mapsto (1 - t)\vec{d} \end{aligned}$$

is the desired deformation-retraction.)

(c) $\Rightarrow$ (a): Again, regard $\pi_1(X, x_0)$ as the set of basepoint-preserving homotopy classes of maps $(S^1, s_0) \to (X, x_0)$. The hypothesis that $\pi_1(X, x_0) = 0$ for all choices of x_0 means that every map $S^1 \to X$ is nullhomotopic through a homotopy that preserves the basepoint of S^1. In particular, this entails (a).

Finally, if X is simply-connected, then it is path-connected and (c) holds. Thus (a) holds, and every map $f : S^1 \to X$ is homotopic to a constant map. And since X is path-connected, all constant maps to X are homotopic.

Conversely, if all maps $S^1 \to X$ are homotopic, then in particular the constant maps are homotopic, so X is path-connected. And since (a) holds, (c) holds as well. Thus X is simply-connected.

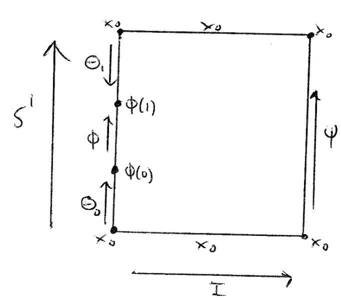
4. (8 marks) Given a space X and a path-connected subspace A containing the basepoint x_0, show that the map $\pi_1(A, x_0) \to \pi_1(X, x_0)$ induced by the inclusion $A \hookrightarrow X$ is surjective iff every path in X with endpoints in A is homotopic to a path in A. Note that whenever Hatcher talks about homotopy of paths he means homotopy relative the endpoints. So the statement actually says "... iff every path in X with endpoints in A is homotopic rel endpoints to a path in A."

Solution

If every path in X with endpoints in A is homotopic to a path in A, then in particular every loop in X with basepoint x_0 is homotopic to a loop in A. This is equivalent to $\pi_1(A, x_0) \to \pi_1(X, x_0)$ being surjective.

Conversely, suppose $\pi_1(A, x_0) \to \pi_1(X, x_0)$ is surjective. And suppose ϕ is a path in X with endpoints in A. Using path-connectivity of A, choose paths θ_0 and θ_1 from x_0 to the points $\phi(0)$ and $\phi(1)$, respectively. Then $\theta_0 \cdot \phi \cdot \bar{\theta_1}$ is a loop with basepoint x_0. By our surjectivity hypothesis, there is a homotopy h from it to a loop ψ in A.

Consider the homotopy of paths $h : I \times I \to X$ from $\theta_0 \cdot \phi \cdot \bar{\theta_1}$ to ψ. The domain $I \times I$ is sketched below. A path traversing the middle third of the left side is homotopic to a path traversing the perimeter counterclosckwise, from $\phi(0)$ to $\phi(1)$. Composing that homotopy with h gives a homotopy of paths from ϕ to $\bar{\theta_0} \cdot x_0 \cdot \psi \cdot x_0 \cdot \theta_1$. (Up to reparametrizing. x_0 here denotes the constant path with value x_0.) Note that the latter path lies in A.



Alternatively one can compose the multiply (concatenate) the homotopy h with the constant homotopy equal to $\bar{\theta_0}$ on the left and with the constant homotopy equal to θ_1 on the right. That will give a homotopy from ϕ to $\bar{\theta_0} \cdot \psi \cdot \theta_1$, a path in A.

5. (2+3+4+4+4+4 marks) Show that there are no retractions $r : X \to A$ in the following cases:

(a) $X = \mathbb{R}^3$ with A any subspace homeomorphic to S^1.
(b) $X = S^1 \times D^2$ with A its boundary torus $S^1 \times S^1$.
(c) $X = S^1 \times D^2$ and A the circle shown in the figure.
(d) $X = D^2 \vee D^2$ with A its boundary $S^1 \vee S^1$.

(e) X a disk with two points on its boundary identified and A its boundary $S^1 \vee S^1$.

(f) X the Möbius band and A its boundary circle.

Solution

Recall that if X retracts onto A, then $\pi_1(A, x_0)$ can be identified with a subgroup of $\pi_1(X, x_0)$, and the retraction induces a homomorphism $\pi_1(X, x_0) \to \pi_1(A, x_0)$ that restricts to the identity on $\pi_1(A, x_0)$.

(a) $\pi_1(\mathbb{R}^3) = 0$ and $\pi_1(A) \cong \pi_1(S^1) \cong \mathbb{Z}$. $\mathbb{Z}$ does not inject into 0.

(b) $\pi_1(S^1 \times D^2) \cong \pi_1(S^1) \cong \mathbb{Z}$ because $S^1 \times D^2$ deformation-retracts onto its core circle. Or because the fundamental group of a product is the product of the fundamental groups. $\pi_1(S^1 \times S^1) \cong \mathbb{Z} \times \mathbb{Z}$. But there is no injective homomorphism from $\mathbb{Z} \times \mathbb{Z}$ into $\mathbb{Z}$ (If this does not seem obvious, think of the least common multiple of two integers).

(c) The inclusion $A \hookrightarrow X$ is nullhomotopic (the image of A is allowed to cross itself during the homotopy), so the induced map $\pi_1(A) \to \pi_1(X)$ is trivial, not injective. Alternatively, you can also see that the induced map on the homotopy groups the generator of $\pi_1(A)$, one lap around that circle, is sent to a loop in X which on the S^1 factor corresponds to doing one lap around and then coming back. This loop is homotopic to the constant loop.

(d) Let a and b denote paths that trace out the two circle summands of A. Consider the retraction of A onto the first circle summand (taking a quotient that reduces one of the circles to a point). It induces a homomorphism

$$\pi_1(A) \to \pi_1(S^1) \cong \mathbb{Z}$$

that sends $[a]$ to $[a]$ and $[b]$ to 1. In particular, $[a]$ is nontrivial in $\pi_1(A)$. In section 1.2 of the text, we will learn that $\pi_1(A)$ is in fact a free group on $[a]$ and $[b]$.

a represents a nontrivial class in $\pi_1(A)$. However, the image in $\pi_1(D^2 \vee D^2)$ is 1 because the circle bounds a disk. (Here we are using exercise 3 in this set.) Therefore $\pi_1(S^1 \vee S^1) \to \pi_1(D^2 \vee D^2)$ is not injective.

(e) Suppose X is obtained from D^2 by gluing two points x_0 and x_1 on its boundary. Draw an arc in the disk from x_0 to x_1. The image of this arc in X is homeomorphic to a circle. D^2 deformation-retracts onto the arc, and so X deformation-retracts onto the circle. Let c denote a path that traces out this circle. It represents a generator of $\pi_1(X) \cong \mathbb{Z}$.

Using our notation from (d), $\pi_1(A)$ has nontrivial classes $[a]$ and $[b]$. Also, the homomorphism induced by retraction

$$\pi_1(A) \to \pi_1(S^1) \cong \mathbb{Z}$$

sends $[a \cdot \bar{b}]$ to $[a]$, so $[a \cdot \bar{b}]$ is a nontrivial element of $\pi_1(A)$.

If we've chosen the directions of the paths a and b correctly, then they are homotopic in X to c. So the homomorphism induced by inclusion

$$\pi_1(A) \to \pi_1(X) \cong \mathbb{Z}$$

sends both $[a]$ and $[b]$ to $[c]$. And therefore it sends $[a \cdot \bar{b}] = [a][b]^{-1}$ to $[c][c]^{-1} = 1$. Thus the homomorphism isn't injective.

(f) The Möbius band deformation retracts onto its core circle, so its fundamental group is isomorphic to $\mathbb{Z}$. The fundamental group of A is also $\mathbb{Z}$. But the map induced by inclusion

$$\pi_1(A) \to \pi_1(X)$$

is multiplication by 2. To see this, let a be a loop that traces out A. Composing this with the deformation-retraction of X onto the core circle, we get a loop that goes *twice* around.

So we can identify $\pi_1(A)$ with the subgroup $2\mathbb{Z} \subset \mathbb{Z} \cong \pi_1(X)$. A retraction $X \to A$ would induce a homomorphism $\mathbb{Z} \to 2\mathbb{Z}$ that restricts to the identity on $2\mathbb{Z}$. But there is no such homomorphism.

Alternatively, the map given by multiplication by 2 for the integers does not have a left inverse which is a group homomorphism.

6. (5 marks) For spaces X and Y with basepoints x_0 and y_0, let $< X, Y >$ denote the set of basepoint-preserving homotopy classes of basepoint-preserving maps $X \to Y$. Show that a basepoint-preserving homotopy equivalence $(X, x_0) \simeq (X', x_0')$ induces a bijection between $< X, Y >$ and $< X', Y >$.

Solution

A basepoint-preserving homotopy equivalence is a based map $f : X \to X'$ such that there is a based map $g : X' \to X$ and basepoint-preserving homotopies $gf \simeq \mathbb{1}_X$ and $fg \simeq \mathbb{1}_{X'}$.

f gives rise to a map $f^* : \langle X', Y\rangle \to \langle X, Y\rangle$ given by $f^*[\psi] = [\psi \circ f]$. Similarly, g gives rise to a map $g^* : \langle X, Y\rangle \to \langle X', Y\rangle$ given by $g^*[\phi] = [\phi \circ g]$.

f^* is well defined. If $[\psi] = [\phi]$, that means there is a basepoint-preserving homotopy H from ψ to ϕ. But then $\psi \circ f$ and $\phi \circ f$ are basepoint-preserving homotopic via $H \circ (f \times 1_I)$. The same argument shows that g^* is well defined.

We show that f^* and g^* are inverse to each other. For $\phi : X \to Y$ a based map, we have $f^*g^*[\phi] = f^*[\phi \circ g] = [\phi \circ g \circ f] = [\phi]$, because given a basepoint preserving homotopy H from gf to $\mathbb{1}_X$ the map $\phi \circ H$ gives a basepoint-preserving homotopy from $\phi \circ g \circ f$ to ϕ. Similarly, for $\psi : X' \to Y$ a based map, we have $g^*f^*[\psi] = [\psi]$.