MATH 215 C
Homework Assignment # 1
Due Thursday, April 15th

Madsen-Tornehave (Problems are at the back of the book.)

p. 252-253 # 9.5, 9.6, 9.8, 9.9

5.) The proof of Sard's theorem given in class only held for the case when the domain and range were of the same dimension. (You can refer to the proof of Prop. 11.7, pages 99-100, in the text.) Explain where the dimension hypothesis is used in a crucial way. The proof also works in one of the cases $n < m$ or $n > m$ (n, m the dimensions of the domain and range, respectively). Which case is it and why? (Note: It turns out that the theorem is false in the other case unless one assumes a higher degree of smoothness (as we are doing in the course). This is quite subtle and not apparent from the proof given.)

6.) Let X, Y be (non-empty) smooth manifolds, X compact, and Y connected. Prove that, if $f : X \to Y$ is a submersion, then it maps *onto* Y. Conclude that there is no submersion from a non-empty compact manifold to $\mathbf{R}^n$.

7.) Suppose $f : X \to Y$ is a smooth map between smooth manifolds and $Z \subset Y$ is a smooth submanifold. Assume that f is transverse to Z. Prove that $W = f^{-1}(Z)$ is a submanifold of X and that $T_x(W) = (df_x)^{-1}(T_{f(x)}Z)), \forall x \in W$. In other words, the tangent space of the inverse image of Z is the inverse image of the tangent space of Z.

8.) Let $X \subset \mathbf{R}^N$ be a smooth submanifold of $\mathbf{R}^N$ so that $0 \notin X$. For any fixed ℓ show that almost every ℓ-dimensional linear subspace $V \subset \mathbf{R}^N$ is transverse to X. (Use any natural parametrization of the set of linear subspaces; the notion of "almost every" should be the same in all cases.)

9.) Let $SL(n)$ denote the subset of $M(n)$ (the set of all real-valued $n \times n$ matrices) with determinant 1. Prove that $SL(n)$ is a submanifold of $M(n)$. (You can identify $M(n)$ with $\mathbf{R}^{n^2}$ in the obvious way.) Prove that the tangent space of $SL(n)$ at the identity matrix can be identified with the $n \times n$ matrices with trace zero.

10.) Prove that the subset of $M(2)$ consisting of 2×2 matrices with rank exactly equal to 1 is a submanifold of dimension 3.