

MATH 215 C
Homework Assignment # 2
Due Thursday, April 22nd

- 1.) Suppose that $f : M \rightarrow Z \subset Y$ is smooth, M with non-empty boundary, Z a submanifold of Y , and Z and Y with no boundary. Prove that if both f and ∂f are transverse to Z , then $W = f^{-1}(Z)$ a manifold with boundary and that $\partial W = W \cap \partial M$. Prove further that the interior of W (i.e. $W - \partial W$) is a submanifold of M and that ∂W is a submanifold of ∂M , both of the same codimension as that of Z in Y .
- 2.) Suppose that $f : M \rightarrow Z \subset Y$ is smooth with ∂M non-empty, Z a closed submanifold of Y and Z and Y both boundaryless. Suppose that ∂f is transverse to Z .
 - a) Show that f is transverse to Z in an open neighborhood of ∂M .
 - b) Prove that there is a map $g : M \rightarrow Z \subset Y$ homotopic to f so that $\partial f = \partial g$ and g is transverse to Z . (You can use the fact that, for any closed subset C of M and an open neighborhood U of C there is a smooth function $\tau : M \rightarrow [0, 1]$ which equals 1 outside U and equals 0 on a (closed) neighborhood of C . (See Appendix A.) Then modify our proof in the boundaryless case, using τ to damp things off near the boundary.)
- 3.)
 - a) Let V be a finite dimensional vector space and let Δ be the diagonal of $V \times V$. For a linear map $T : V \rightarrow V$, show that the graph of T , as a subset of $V \times V$, is transverse to Δ if and only if $+1$ is not an eigenvalue of T .
 - b) Let $f : X \rightarrow X$ be a smooth map of a manifold to itself and suppose that $x \in X$ is a fixed point of f . Then x is called a Lefschetz fixed point if df_x doesn't have $+1$ as an eigenvalue; f is called a Lefschetz map if all of its fixed points are Lefschetz. Prove that if X is compact and f is Lefschetz, then it has only finitely many fixed points.
- 4.) Let N be a smooth manifold, $\dim N > 0$ and P a closed submanifold of N . Suppose that X is compact, $f : X \rightarrow N$ smooth, and $\dim X + \dim P = \dim N$. Then f is homotopic to g where g is transverse to P . As explained in class, $g^{-1}(P)$ consists of a finite number of points; this number (mod 2) is invariant under homotopy and is called the (mod 2) intersection number of f with P .
 - a) Suppose N is contractible (smoothly). Prove that, for any compact X and any $f : X \rightarrow N$ smooth, the mod 2 intersection number of f with P is 0.
 - b) Prove that no compact manifold, other than a single point, is contractible.
- 5.)
 - a) Let $Z \subset S^n$ be a closed submanifold of dimension $0 < k < n$ and let X be a compact manifold of dimension $n - k$. Prove that for any smooth map $f : X \rightarrow Z \subset S^n$ the mod 2 intersection number of f with Z is 0.
 - b) Prove that the 2-sphere S^2 and the 2-dimensional torus T^2 are not diffeomorphic.

- 6.) Prove that every simply connected smooth manifold is orientable.
- 7.) If V, W are finite dimensional vector spaces, oriented so that $\alpha = \{v_1, v_2, \dots, v_k\}$ and $\beta = \{w_1, w_2, \dots, w_l\}$ are positive bases of V and W , respectively. The *product orientation* of $V \times W$ is defined to be that where the basis $\{(\alpha, 0), (0, \beta)\}$ (i.e. $\{(v_1, 0), (v_2, 0), \dots, (v_k, 0), (0, w_1), \dots, (0, w_l)\}$) is positive.
- Show that, if X is an orientable manifold, then the product orientation on $X \times X$ is the same, independent of which orientation on X is chosen.
 - Show that, if X is *not* orientable, then there is a neighborhood of the diagonal in $X \times X$ which is orientable even though neither the diagonal nor all of $X \times X$ is orientable.
- 8.) Let M, N be compact, oriented submanifolds of \mathbf{R}^k of dimension m, n respectively. Assume that they are disjoint and that $m + n = k - 1$. Define the *linking number*, $Lk(M, N)$, to be the degree of the map

$$\Phi : M \times N \rightarrow S^{k-1}$$

defined by $\Phi(x, y) = (x - y) / \|x - y\|$.

- Prove that $Lk(M, N) = (-1)^{(m-1)(n-1)} Lk(N, M)$.
- Prove that if M bounds a compact, oriented submanifold in $\mathbf{R}^k - N$ or if M is homotopic in $\mathbf{R}^k - N$ to a point, then $Lk(M, N) = 0$.