MATH 215 C
Homework Assignment # 3
Due Thursday, May 6th

Madsen-Tornehave (Problems are at the back of the book.)

p. 253, 259 # 9.14, 11.5

1.) Suppose that X is a compact, non-orientable manifold. Let $\Delta \subset X \times X$ be the diagonal. Use problem 7 from the previous homework assignment to show that $I(\Delta, \Delta)$ (*integer*-valued intersection number) is well-defined even in this case. (We will see that this self-intersection number equals the Euler characteristic of X.)

2.) Let $N(\Delta, X \times X)$ be the normal bundle of the diagonal $\Delta \subset X \times X$, where X is compact and oriented. Prove that, at each point $(x, x) \in \Delta$, the fiber can be represented by vectors of the form $(v, -v) \in T_xX \times T_xX$. Prove that the map $TX \to N(\Delta, X \times X)$ defined by $(x, v) \to ((x, x), (v, -v))$ is a diffeomorphism. Use this to prove that there is diffeomorphism from a neighborhood of the zero section Z of TX to a neighborhood of Δ in $X \times X$, extending the map from Z to Δ. Explain why this implies that the index of any non-degenerate vector field equals $I(\Delta, \Delta)$.

3.) Let J, K be two disjoint oriented knots in S^3. Consider a regular projection of the link $J \cup K$ (i.e. the only crossings and self-crossings are transverse double points). Define the linking number $lk(J, K)$ to be the sum of $sgn\ (x_i)$ where the sum is over the set of points where J crosses *under* K and the sign is computed as on page 106 of the text. Prove that this linking number is the same as the one defined in Problem 8 of the previous problem set.

Note: A version of this fact is discussed in the text, p. 104-106. You may want to consult that for reference. However, you will need to adapt it somewhat since, in that section, the text is using the definition of degree coming from deRham cohomology (which we will get to later in the quarter).

4.) The *Hopf invariant*, $H(f)$, of $f : S^3 \to S^2$ is $Lk(f^{-1}(a), f^{-1}(b))$, where a, b are distinct regular values of f and the linking number is computed in $\mathbf{R}^3 = S^3 - c$ for any c such that $f(c) \neq a, b$. Prove that:

a) $H(f)$ is a homotopy invariant which vanishes if f is null-homotopic.

b) If $g : S^3 \to S^3$ has degree p, then $H(fg) = pH(f)$.

c) If $h : S^2 \to S^2$ has degree q, then $H(hf) = q^2H(f)$.

d) Consider S^3 as unit vectors in $\mathbf{C}^2$ and S^2 as complex projective plane, and define $f(z, w) = [z, w]$. (This is called the *Hopf map.*) Prove that $|H(f)| = 1$; hence f is not null-homotopic.

5.)

a) Prove that the (open) Mobius strip is a 1-dimensional vector bundle over the circle (use the "core curve" as the circle).

b) Show that the (mod 2) self-intersection number of the zero section is non-zero. Use this to prove that the bundle is not a product.

6.) Let E be an oriented k-dimensional vector bundle over an oriented n-dimensional manifold M, $k \leq n$. Let $s : M \to E$ be a section of this bundle which is transverse to the zero section $Z \subset E$. Prove that $W = s^{-1}(Z)$ determines an $n - k$-dimensional cycle in M, hence an element of $H_{n-k}(M)$. Show that any two such sections determine the same homology class.

You may assume that any smooth, oriented manifold (W in particular) can be given an oriented triangulation. (This makes the first part of the problem pretty immediate.)