CME 303/MATH 220: PROBLEM SET 6
DUE THURSDAY, NOVEMBER 15, 2018

Problem 1.

(i) On $\mathbb{R}^3$, find the Fourier transform of the function $g(x) = |x|^{-1}$. (Hint: to do this efficiently, consider $g(x)$ as the limit of $g_a(x) = e^{-a|x|}|x|^{-1}$, and use your result from the last problem set.)

(ii) Solve $\Delta u = f$ on $\mathbb{R}^3$, where $f \in \mathcal{S}(\mathbb{R}^3)$, writing your answer as a convolution.

Problem 2. Find the Fourier transform on $\mathbb{R}^3$ of the distribution $u = \delta_{|x|-R}$, i.e. for $\psi \in \mathcal{S}(\mathbb{R}^3)$,

$$u(\psi) = R^2 \int_{\mathbb{S}^2} \psi(R\omega)\, dS(\omega),$$

or in spherical coordinates

$$u(\psi) = R^2 \int_0^{\pi} \int_0^{2\pi} \psi(R\cos\theta, R\sin\theta\cos\phi, R\sin\theta\sin\phi)\, \sin\theta\, d\phi\, d\theta.$$

(*Hint*: use that u is compactly supported, so you can evaluate the Fourier transform directly by applying it to $e^{-ix\cdot\xi}$, and use spherical coordinates centered around ξ.)

Problem 3. Write the solution of the wave equation on $\mathbb{R}^3_x \times \mathbb{R}_t$,

$$u_{tt} = c^2 \Delta_x u,\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x),$$

in a form that does not involve the Fourier transform by using convolutions.

Hint: Use the previous problem to deal with the ψ-term. To deal with the ϕ term, note that

$$\cos(c|\xi|t) = \frac{\partial}{\partial t}\left(\frac{\sin(c|\xi|t)}{c|\xi|}\right).$$

Problem 4. Show that if $u \in \mathcal{S}'(\mathbb{R}^n)$ then there is an integer $m \geq 0$ and $C > 0$ such that for all $\phi \in \mathcal{S}(\mathbb{R}^n)$,

$$|u(\phi)| \leq C\|\phi\|_m$$

where

$$\|\phi\|_m = \sum_{|\alpha|\leq m, |\beta|\leq m} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial_x^\beta \phi|.$$

Hints: This relies on the continuity of u as a map $u : \mathcal{S} \to \mathbb{C}$. So suppose for the sake of contradiction that no such m and C exist; in particular for an integer $j > 0$, $m = j$ and $C = j$ do not work, i.e. there exists $\phi_j \in \mathcal{S}$ such that

$$|u(\phi_j)| > j\|\phi_j\|_j.$$

Note that ϕ_j cannot be 0 (for then $u(\phi_j)$ would vanish by linearity). Let $\psi_j = \frac{1}{j\|\phi_j\|_j}\phi_j$, so $\psi_j \in \mathcal{S}$, $\|\psi_j\|_j = \frac{1}{j}$ and

$$|u(\psi_j)| > j\|\psi_j\|_j = 1.$$

Now show that $\psi_j \to 0$ in $\mathcal{S}$ as $j \to \infty$, and use this to get a contradiction with the continuity of u.

Problem 5. Suppose that $u \in \mathcal{S}'(\mathbb{R}^n)$ and u has compact support. (This means that there is a function $f \in C_c^\infty(\mathbb{R}^n)$ such that $u = fu$; namely one would take f identically 1 on a neighborhood of the support of u.) Show that that the C^∞ function $\mathcal{F}u$ satisfies

$$|(\mathcal{F}u)(\xi)| \leq C(1 + |\xi|)^m$$

for some C and m.

Hint: Use the result of the previous problem and that $(\mathcal{F}u)(\xi) = u(\phi_\xi)$, $\phi_\xi(x) = f(x)e^{-ix\cdot\xi}$.

Problem 6. Suppose that $u, v \in \mathcal{S}'(\mathbb{R}^n)$ and v has compact support. Give a definition of $u * v \in \mathcal{S}'(\mathbb{R}^n)$ that is consistent with the definition if one of the two distributions is in $\mathcal{S}(\mathbb{R}^n)$.

Problem 7. Suppose that $P = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ on $\mathbb{R}^n$. A *fundamental solution* for P is a distribution E such that $PE = \delta_0$. E is also called a *Green's function* with pole at 0.

(i) Show that if E is a fundamental solution for P then for $f \in C_c^\infty(\mathbb{R}^n)$, $u = E * f$ solves $Pu = f$.

(ii) Show that the same holds even if $f \in C_c^0(\mathbb{R}^n)$ (or indeed $f \in \mathcal{D}'(\mathbb{R}^n)$ with compact support). You may assume that $E \in \mathcal{S}'(\mathbb{R}^n)$ to use your results from the previous problem set, if you wish (though this is not strictly necessary).

(iii) Show that the distribution E given by the function $\frac{-1}{4\pi|x|}$ is a fundamental solution for Δ in $\mathbb{R}^3$ in two different ways: using the Fourier transform, and directly.

Hint: For the direct calculation, to find ΔE, recall that $\Delta E(\phi) = E(\Delta\phi)$, and write the right hand side as $-\lim_{\epsilon\to 0}\int_{|x|\geq\epsilon}\frac{1}{4\pi|x|}\Delta\phi(x)\,dx$, and use the divergence theorem (integrate by parts in polar coordinates).

Problem 8. *(Optional!)* Show that every $u \in \mathcal{S}'(\mathbb{R}^n)$ can be approximated by elements of $\mathcal{S}(\mathbb{R}^n)$, i.e. show that there exist $f_j \in \mathcal{S}(\mathbb{R}^n)$ such that $\iota_{f_j} \to u$ in $\mathcal{S}'(\mathbb{R}^n)$. Here recall that $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ means that $u_j(\phi) \to u(\phi)$ for all $\phi \in \mathcal{S}(\mathbb{R}^n)$.

Hints: it suffices to show that there is f_j and m such that $|\iota_{f_j}(\phi) - u(\phi)| \leq j^{-1}\|\phi\|_m$ for all $\phi \in \mathcal{S}$, see Problem 4 for the notation. So first consider $v_j = \chi_j u$, where $\chi_j(x) = \chi(x/j)$, and $\chi \in C_c^\infty(\mathbb{R}^n)$ is identically 1 for $|x| < 1$, identically 0 for $|x| > 2$. Thus, v_j are compactly supported distributions, and show that $v_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ in the strong sense that for some $\tilde{m}$, $|v_j(\phi) - u(\phi)| \leq \tilde{C}j^{-1}\|\phi\|_{\tilde{m}}$. Now to get the f_j, we approximate the v_j as follows: $\mathcal{F}v_j$ is C^∞, but does not decay at infinity. So let $g_{jk}(\xi) = \chi_k(\xi)(\mathcal{F}v_j)(\xi)$, χ_j as above. Show that $g_{jk} \in \mathcal{S}(\mathbb{R}^n)$, and $\iota_{g_{jk}} \to \mathcal{F}v_j$ in $\mathcal{S}'(\mathbb{R}^n)$ as $k \to \infty$. Now consider $\mathcal{F}^{-1}g_{jk}$.