CME 303/MATH 220: PROBLEM SET 7
DUE 10AM, FRIDAY, NOVEMBER 30, 2018

Problem 1. On $\mathbb{R}^3$, write $x = (x', x_3)$, so $x' = (x_1, x_2)$. Suppose f is a compactly supported C^1 function in $x_3 \geq 0$ in $\mathbb{R}^3$ vanishing near $x_3 = 0$. Find the solution u of

$$\Delta u = f,\ x_3 \geq 0,$$
$$\partial_{x_3} u(x', 0) = 0$$

which goes to 0 at infinity. Write your solution as explicitly in terms of f as possible.

Problem 2. (i) Using the method of reflection, solve the wave equation with Neumann boundary conditions on the interval $[0, \ell]_x$:

$$u_{tt} - c^2 u_{xx} = 0,\ u(x, 0) = \phi(x),\ u_t(x, 0) = \psi(x),\ u_x(0, t) = 0 = u_x(\ell, t).$$

You do not need to write an explicit formula containing only ϕ and ψ; the appropriate extension of ϕ and ψ to $\mathbb{R}$ may appear in the formula.

(ii) If $\psi = 0$ and ϕ is C^∞ except at a point $x_0 \in (0, \ell)$, where do you know for sure that u is C^∞?

Problem 3. Solve the inhomogeneous heat equation on the half-line for Dirichlet boundary conditions:

$$u_t - k u_{xx} = f,\ u(x, 0) = \phi(x),\ u(0, t) = 0,$$

in two different ways:

(i) Using Duhamel's principle, and the solution formula for the homogeneous equation derived in class (i.e. with $f = 0$) on the half line.

(ii) Using the appropriate extension of f and ϕ to the whole real line and solving the inhomogeneous PDE on the real line.

Problem 4. Derive Duhamel's principle for the wave equation on $\mathbb{R}$

$$u_{tt} - c^2 \partial_x^2 u = f,\ u(x, 0) = \phi(x),\ u_t(x, 0) = \psi(x),$$

by setting up a first order system for $U = \begin{bmatrix} u \\ v \end{bmatrix}$, $v = u_t$, namely

$$u_t - v = 0,\ u(x, 0) = \phi(x),$$
$$v_t - c^2 \partial_x^2 u = f,\ v(x, 0) = \psi(x).$$

Thus, one has

$$\partial_t U - AU = \begin{bmatrix} 0 \\ f \end{bmatrix},\ U(0, x) = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix},$$

where

$$A = \begin{bmatrix} 0 & \mathrm{Id} \\ c^2 \partial_x^2 & 0 \end{bmatrix}.$$

This is now a first order equation in time, so Duhamel's principle for first order equations is applicable, and gives the solution of the inhomogeneous equation as

$$U(x, t) = \mathcal{S}(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix}(x) + \int_0^t \mathcal{S}(t - s) \begin{bmatrix} 0 \\ f_s \end{bmatrix}(x)\, ds,$$

where $\mathcal{S}$ is the solution operator for the homogeneous problem $\partial_t U - AU = 0$. You need to work this out explicitly, in particular what $\mathcal{S}$ is, to derive the solution of the wave equation.

Problem 5. (i) Consider the following eigenvalue problem on $[0, \ell]$:

$$-X'' = \lambda X,\ X(0) = 0,\ X'(\ell) = 0.$$

Find all eigenvalues and eigenfunctions.

(ii) Using separation of variables, find the general 'separated' solution of the wave equation

$$u_{tt} = c^2 u_{xx},\ u(0,t) = 0,\ u_x(\ell, t) = 0.$$

(iii) Solve the wave equation with initial conditions

$$u(x, 0) = \sin(3\pi x/(2\ell)) - 2\sin(5\pi x/(2\ell)),\ u_t(x, 0) = 0.$$

(iv) Using separation of variables, find the general 'separated' solution of the *heat equation*

$$u_t = k u_{xx},\ u(0,t) = 0,\ u_x(\ell, t) = 0,$$

here $k > 0$ constant.

Problem 6. Consider the wave equation on a ring of length 2ℓ. We let x be the arclength variable along the ring, $x \in [-\ell, \ell]$. We would like to understand wave propagation along the ring, so consider the wave equation with *periodic boundary conditions*:

$$u_{tt} = c^2 u_{xx},\ u(-\ell, t) = u(\ell, t),\ u_x(-\ell, t) = u_x(\ell, t).$$

(i) Find the general 'separated' solution.

(ii) Find the solution with initial condition

$$u(x, 0) = 0,\ u_t(x, 0) = \cos(2\pi x/\ell) - \sin(\pi x/\ell),\ x \in [-\ell, \ell].$$

(iii) Give an alternative method of solution by extending u to be a 2ℓ-periodic function in x on all of $\mathbb{R}$, and using d'Alembert's formula.

(iv) How do singularities of u propagate? That is, if the only singularity of the initial data is at some x_0 (i.e. they are C^∞ elsewhere), where can u be singular? Interpret this physically.