CME 303/MATH 220: MIDTERM
NOVEMBER 1, 2018

This is a closed book, closed notes, no electronic devices exam.

There are 5 problems. **Solve Problems 1-3 and one of Problems 4 and 5.** Write your solutions to problems 1 and 2 in blue book #1, and your solutions to problems 3, 4 and 5 in blue book #2. Within each book, you may solve the problems in any order. Total score: 100 points.

Problem 1. *(25 points)*

(i) Solve the PDE
$$(x-1)u_x + u_y = 2u, \ u(0,y) = e^y,$$
on as large of a subset of $\mathbb{R}^2$ containing the y-axis as you can. Do all characteristics intersect the y-axis, and do so in a unique point, non-tangentially? What does this mean for the solutions of the initial value problem of the PDE? Sketch the characteristics.

(ii) Regarding the region of solvability and the characteristics mentioned in part (i), what changes if the right hand side of the PDE is replaced by u^2:
$$(x-1)u_x + u_y = u^2, \ u(0,y) = e^y \ ?$$
You do not need to give a full solution of the PDE, but you need to justify your answer.

Problem 2.

(i) *(7 points)* Find the general C^2 solution of the PDE
$$u_{xx} - 5u_{xt} + 4u_{tt} = 0.$$

(ii) *(7 points)* Solve the initial value problem with initial condition
$$u(x,0) = \phi(x), \ u_t(x,0) = \psi(x),$$
with ϕ, ψ given C^2 functions.

(iii) *(6 points)* If $\phi(x)$ and $\psi(x)$ are 0 for $|x| \geq 1$, where can you say for sure that u is 0? Explain the geometric meaning.

(iv) *(5 points)* Give an example of a discontinuous distributional solution of the PDE, and explain why this solves the PDE (you may use results from the problem sets).

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Problem 3. *(25 points)* Consider the (real-valued) heat equation on $[0,\ell]_x \times [0,\infty)_t$ with Dirichlet boundary conditions:

$$u_t = (k(x)u_x)_x, \; u(0,t) = 0 = u(\ell,t),$$

where $k > 0$ is allowed to depend on x, $k \in C^1([0,\ell])$. Assume throughout that u is C^2. Let

$$E(t) = \frac{1}{2}\int_0^\ell (u(x,t)^2 + k(x)u_x(x,t)^2)\,dx.$$

(i) Show that E is a decreasing function of t: $E(t_1) \geq E(t_2)$ if $t_1 \leq t_2$.
(ii) Show the solution of the variable coefficient heat equation (under the conditions mentioned above) with given initial condition, $u(x,0) = \phi(x)$, is unique.
(iii) Give a stability statement for the solution of the variable coefficient heat equation.
(iv) Are your conclusions also valid for Neumann boundary conditions $u_x(0,t) = 0 = u_x(\ell,t)$ in place of the Dirichlet boundary conditions?

Solve one of Problems 4 and 5.

Problem 4. Consider the equation

$$u_t + uu_x = 0, \; u(x,0) = \phi(x), \; t \geq 0.$$

(i) *(6 points)* State the definition of u being a weak solution of this PDE.
(ii) *(10 points)* Suppose $c \in \mathbb{R}$, and $u(x,t) = 1$ if $x < ct$ and $u(x,t) = 0$ if $x > ct$. For what value of c does this give a weak solution of the PDE (for suitable initial condition)? Derive this directly from the definition of a weak solution, i.e. derive the Rankine-Hugoniot condition in this case.
(iii) *(9 points)* Suppose the initial condition is

$$\phi(x) = \begin{cases} 1, & x < 0, \\ 1-x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Will the solution develop a shock in $t > 0$? Why/why not?

Problem 5.

(i) *(5 points)* State the definition of a function f being Schwartz: $f \in \mathcal{S}(\mathbb{R}^n)$.
(ii) *(7 points)* Show that for f Schwartz, the Fourier transform of $f_a(x) = f(x-a)$ is $e^{-ia\cdot\xi}(\mathcal{F}f)(\xi)$, and use this to show that the Fourier transform of $D_j f = \frac{1}{i}\partial_j f$ is $\xi_j(\mathcal{F}f)(\xi)$.
(iii) *(7 points)* On $\mathbb{R}$, use the Fourier transform (by taking the F.T. of both sides) to solve the ODE

$$-\frac{d^2}{dx^2}u + u = e^{-x^2/2}.$$

You may leave your answer as the inverse Fourier transform of a function. You may use that the Fourier transform of $f(x) = e^{-ax^2}$, $a > 0$, is $(\pi/a)^{1/2}e^{-\xi^2/(4a)}$.
(iv) *(6 points)* Is the solution of the ODE obtained by the Fourier transform unique? How do you reconcile this with the fact that the ODE existence and uniqueness theorem guarantees at least local solvability and uniqueness to the initial value problem, where one also imposes $u(0) = c_0$, $u'(0) = c_1$, with $c_0, c_1 \in \mathbb{C}$ arbitrary? (*Hint:* what are the solutions of the homogeneous ODE?)