MATH 220: FINAL EXAM – DECEMBER 11, 2009

This is a closed book, closed notes, no calculators exam. There are 7 problems. Solve all of them. Write your solutions to problems 1, 2 and 3 in blue book(s) #1, and your solutions to problems 4, 5, 6 and 7 in blue book(s) #2. Within each book, you may solve the problems in any order. Total score: 200 points.

Use blue book(s) #1 for Problems 1-3!

Problem 1. (i) *(20 points)* For $|y-1|$ small, solve

$$xu_x + yu_y = 1,\ u(x,1) = x^2.$$

Sketch the characteristics, and discuss where in $\mathbb{R}^2$ is the solution uniquely determined by the initial data. Does the solution you found extend to this region? Does it extend to a larger region?

(ii) *(15 points)* For $|x|$ small, solve

$$uu_x + uu_y = 1,\ u(0,y) = y^2 + 1.$$

Problem 2. Consider the wave equation $u_{tt} = c^2 u_{xx}$ on the half-line, i.e. on $[0,\infty)_x \times [0,\infty)_t$, with homogeneous Dirichlet boundary condition $u(0,t) = 0$, and with initial conditions $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$ for $x \geq 0$.

(i) *(10 points)* Find u.

(ii) *(8 points)* Suppose ϕ, ψ are both linear near 0 (i.e. $\phi(x) = cx$ for x small, and similarly for ψ), and are C^∞ away from a point $x_0 > 0$. Where can you say for sure that u is C^∞?

(iii) *(7 points)* Suppose that $\phi \equiv 0$, and $\psi(x) = x$ for $x < 1$, $\psi(x) = 0$ for $x > 1$. Find $u(x,t)$ explicitly for $t \geq 0$. (Hint: it is best to consider different cases depending on where (x,t) lies.) Does the location of the singularities (lack of being C^∞) agree with what you found in (ii)?

You may use in any part of the problem that if v solves $v_{tt} - c^2 v_{xx} = 0$ on $\mathbb{R}^2$ then

$$v(x,t) = \frac{v(x-ct,0) + v(x+ct,0)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} v_t(x',0)\,dx'$$

Problem 3. *(25 points)* Consider the (real-valued) damped wave equation on $[0,\ell]_x \times [0,\infty)_t$ with *Robin boundary conditions*:

$$u_{tt} + a(x)u_t = (c(x)^2 u_x)_x,\ u_x(0,t) = \alpha u(0,t),\ u_x(\ell,t) = -\beta u(\ell,t)$$

where $\alpha, \beta \geq 0$ are constants, $a \geq 0$ and $c > 0$ depend on x only, and there are constants $c_1, c_2 > 0$ such that $c_1 \leq c(x) \leq c_2$ for all x. (Note that if $\alpha = 0$ and $\beta = 0$ then this is just the Neumann boundary condition! In general, this BC would hold for example for a string if its ends were attached to springs.) Assume throughout that u is C^2. Let

$$E(t) = \frac{1}{2}\int_0^\ell (u_t(x,t)^2 + c(x)^2 u_x(x,t)^2)\,dx + \frac{1}{2}(c(0)^2\alpha u(0,t)^2 + c(\ell)^2\beta u(\ell,t)^2).$$

(i) Show that if $a \equiv 0$ then E is constant.

(ii) Show that if $a \geq 0$ then E is a decreasing (i.e. non-increasing) function of t, and that the solution of the damped wave equation (under the conditions mentioned above) with given initial condition is unique.

Use blue book(s) #2 for Problems 4-7!

Problem 4. (i) *(8 points)* Consider the following eigenvalue problem on $[0, \ell]$:

$$-X'' = \lambda X,\ X(0) = 0,\ X'(\ell) = 0.$$

Find all eigenvalues and eigenfunctions, and show that eigenfunctions corresponding different eigenvalues are orthogonal to each other.

(ii) *(8 points)* Using separation of variables, find the general 'separated' solution of the heat equation (with $k > 0$ fixed):

$$u_t = ku_{xx},\ u(0,t) = 0,\ u_x(\ell, t) = 0.$$

(iii) *(6 points)* Solve the heat equation with initial condition

$$u(x, 0) = \phi(x),$$

i.e. give a formula for the series coefficients in part (ii) in terms of ϕ.

(iv) *(8 points)* Now suppose $\phi(x) = x(\ell - x)^2$. Give an estimate for the coefficients in the series which implies the uniform convergence of the series on $[0, \ell] \times [0, \infty)_t$, and explain how the estimate implies uniform convergence. You do not need to compute the coefficients, though that is one way of getting the desired estimate.

Problem 5. (i) *(15 points)* For both of the following functions f on $[0, \ell]$, state whether the Fourier sine series on $[0, \ell]$ converges in each of the following senses: uniformly, in L^2. State what the Fourier series converges to on all of $\mathbb{R}$. Make sure that you give the reasoning that led you to the conclusions.

(a) $f(x) = x^2(\ell - x)^4$,

(b) $f(x) = 0$, for $0 \leq x \leq \ell/2$, and $f(x) = x - \ell/2$ for $\ell/2 < x \leq \ell$.

(ii) *(10 points)* For the function f in (b) above, we wish to approximate f by a function g of the form $a_1 \sin(\pi x/\ell) + a_3 \sin(3\pi x/\ell)$ on $[0, \ell]$. Find the constants a_1 and a_3 that minimize the L^2 error, $\int_0^\ell |f - g|^2\, dx$, of the approximation.

Problem 6. Recall that $\mathcal{S}(\mathbb{R}^n)$ is the set of Schwartz functions on $\mathbb{R}^n$.

(i) *(7 points)* Show that if $\phi, \psi \in C^0(\mathbb{R}^n)$ with $(1+|x|)^N \phi(x)$, $(1+|x|)^N \psi(x)$ both bounded for some $N > n$ then

$$\int_{\mathbb{R}^n} (\mathcal{F}\phi)(\xi)\, \psi(\xi)\, d\xi = \int_{\mathbb{R}^n} \phi(x)\, (\mathcal{F}\psi)(x)\, dx.$$

(ii) *(6 points)* Define $\mathcal{F}u$ if u is a tempered distribution, i.e. $u \in \mathcal{S}'(\mathbb{R}^n)$, and show that this is consistent with the standard definition if $u = \iota_\phi$, $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$.

(iii) *(5 points)* Recall that $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ means that for each $\phi \in \mathcal{S}(\mathbb{R}^n)$, $u_j(\phi) \to u(\phi)$. Show that if $u_j \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ then $\mathcal{F}u_j \to \mathcal{F}u$ in $\mathcal{S}'(\mathbb{R}^n)$.

(iv) *(5 points)* Show that for $\phi \in C^0(\mathbb{R}^n)$ with $(1 + |x|)^N \phi(x)$ bounded for some $N > n$,

$$\overline{\mathcal{F}\phi(\xi)} = (2\pi)^n (\mathcal{F}^{-1}\overline{\phi})(\xi).$$

(v) *(7 points)* Show the *Parseval/Plancherel formula*, i.e. that for $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \phi(x)\, \overline{\psi(x)}\, dx = (2\pi)^{-n} \int_{\mathbb{R}^n} (\mathcal{F}\phi)(\xi)\, \overline{(\mathcal{F}\psi)(\xi)}\, d\xi,$$

and hence conclude that, up to a constant factor, the Fourier transform preserves L^2-norms:

$$\|\mathcal{F}\phi\|_{L^2(\mathbb{R}^n)} = (2\pi)^{n/2} \|\phi\|_{L^2(\mathbb{R}^n)}.$$

Problem 7. In $\mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_t$, we write points as $(x_1, \dots, x_n, t)$, and also write $x = (x_1, \dots, x_n)$. With $\Delta_x = \sum_{j=1}^n \frac{\partial^2}{\partial x_j^2}$, consider the modified wave equation in $\mathbb{R}^{n+1}$:

$$u_{tt} - c^2 \Delta_x u - \lambda u = f. \tag{1}$$

(i) *(12 points)* Show that if $f \in \mathcal{S}(\mathbb{R}^{n+1})$, then (1) has a unique solution u in $\mathcal{S}(\mathbb{R}^{n+1})$ when $\operatorname{Im} \lambda \neq 0$, and give an expression for u in terms of f. Your final formula may involve the (inverse) Fourier transform. (Hint: use the Fourier transform in all variables!)

(ii) *(12 points)* Still assuming $\operatorname{Im} \lambda \neq 0$, show that if $\phi, \psi \in \mathcal{S}(\mathbb{R}^n)$, $f \equiv 0$ then the PDE (1) together with the initial conditions

$$u(x, 0) = \phi(x),\ u_t(x, 0) = \psi(x),$$

has a unique solution which is bounded as long as t varies in bounded intervals. Again, give an expression for u in terms of ϕ, ψ. Your final formula may involve the (inverse) partial Fourier transform.

(iii) *(6 points)* Compare (i) and (ii): in (ii) we impose an arbitrary additional condition: why does this not violate the uniqueness of (i) (note that for different ϕ the solutions are certainly different!)?