1 Introduction

1.1 Basic Definitions and Examples

Let u be a function of several variables, $u(x_1, \ldots, x_n)$. We denote its partial derivative with respect to x_i as

$$u_{x_i} = \frac{\partial u}{\partial x_i}.$$

For short-hand notation, we will sometimes write the partial differential operator $\frac{\partial}{\partial x_i}$ as ∂_{x_i}. With this notation, we can also express higher-order derivatives of a function u. For example, for a function $u = u(x, y, z)$, we can express the second partial derivative with respect to x and then y as

$$u_{xy} = \frac{\partial^2 u}{\partial_y \partial_x} = \partial_y \partial_x u.$$

As you will recall, for "nice" functions u, mixed partial derivatives are equal. That is, $u_{xy} = u_{yx}$, etc. See Clairaut's Theorem.

In order to express higher-order derivatives more efficiently, we introduce the following multi-index notation. A **multi-index** is a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ where each α_i is a nonnegative integer. The **order of the multi-index** is $|\alpha| = \alpha_1 + \ldots + \alpha_n$. Given a multi-index α, we define

$$D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} = \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} u. \tag{1.1}$$

Example 1. Let $u = u(x, y)$ be a function of two real variables. Let $\alpha = (1, 2)$. Then α is a multi-index of order 3 and

$$D^\alpha u = \partial_x \partial_y^2 u = u_{xyy}.$$

If k is a nonnegative integer, we define

$$D^k u = \{D^\alpha u \,:\, |\alpha| = k\}, \tag{1.2}$$

the collection of all partial derivatives of order k.

Example 2. If $u = u(x_1, \ldots, x_n)$, then $D^1 u = \{u_{x_i} \,:\, i = 1, \ldots, n\}$.

With this notation, we are now ready to define a partial differential equation. A **partial differential equation** is an equation involving a function u of several variables and its partial derivatives. The **order** of the partial differential equation is the order of the highest-order derivative that appears in the equation.

Example 3.

- $u_t = u_x$ (Transport Eqn., first order)
- $u_t = k u_{xx}$ (Heat Eqn., second order)
- $u_{tt} = c^2 u_{xx}$ (Wave Eqn., second order)

- $u_{xx} + u_{yy} = 0$ (Laplace Eqn., second order)
- $u_{xx} + u_{yy} = f(x, y)$ (Poisson's Eqn., second order)
- $u_t + u_{xxx} + uu_x = 0$ (KdV Eqn., third order)
- $u_x^2 + u_y^2 = c^2$ (Eikonal Eqn. of Geometric Optics, first order)

In general, a k-th order partial differential equation is an equation which can be written in the form

$$F(\vec{x}, u, Du, D^2u, \ldots, D^ku) = 0$$

for some function F. For example, for the transport equation, the associated function $F : \mathbb{R}^5 \to \mathbb{R}$ is given by $F(x, t, u, p, q) = q - p$. In this way, the transport equation can be written as

$$F(x, t, u, u_x, u_t) = u_t - u_x = 0.$$

There are a wide variety of partial differential equations and, consequently, we cannot hope to look for a general method for solving all of them. Instead, we will try to solve some of the "easier" ones and continue from there. So, what do we mean by "easier"? We will begin by introducing some definitions to classify partial differential equations into different categories. In particular, we begin by introducing the concept of *linearity*.

A k-th order PDE is **linear** if it can be written as

$$\sum_{|\alpha| \leq k} a_\alpha(\vec{x}) D^\alpha u = f(\vec{x}). \tag{1.3}$$

If $f = 0$, the PDE is **homogeneous**. If $f \neq 0$, the PDE is **inhomogeneous**. If it is not linear, we say it is **nonlinear**.

Example 4.

- $u_t + u_x = 0$ is homogeneous linear
- $u_{xx} + u_{yy} = 0$ is homogeneous linear.
- $u_{xx} + u_{yy} = x^2 + y^2$ is inhomogeneous linear.
- $u_t + x^2 u_x = 0$ is homogeneous linear.
- $u_t + u_{xxx} + uu_x = 0$ is not linear.
- $u_x^2 + u_y^2 = 1$ is not linear.

As you may be able to guess, many equations are not linear. In studying partial differential equations, it is sometimes easier to distinguish further among nonlinear equations. We will do so by introducing the following definitions.

We say a k-th-order nonlinear partial differential equation is **semilinear** if it can be written in the form

$$\sum_{|\alpha| = k} a_\alpha(\vec{x}) D^\alpha u + a_0(D^{k-1}u, D^{k-2}u, \ldots, Du, u, \vec{x}) = 0. \tag{1.4}$$

In particular, this means that semilinear equations are ones in which the coefficients of the terms involving the highest-order derivatives of u depend only on x, not on u or its derivatives.

Example 5.

- $u_t + u_x + u^2 = 0$ is semilinear.
- $u_t + u_{xxx} + uu_x = 0$ is semilinear.
- $u_t + xu_x = 0$ is linear.
- $u_t + uu_x = 0$ is not semilinear.

We say a k-th-order nonlinear partial differential equation, which is not semilinear, is **quasilinear** if it can be written in the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u, \ldots, Du, u, \vec{x})D^\alpha u + a_0(D^{k-1}u, \ldots, Du, u, \vec{x}) = 0. \tag{1.5}$$

In particular, this means that quasilinear equations are those equations in which the coefficients of the highest-order terms may depend on $\vec{x}, u, \ldots, D^{k-1}u$, but not on $D^k u$.

Example 6.

- $u_t + uu_x = 0$ is quasilinear.
- $u_t + a(u)u_x = 0$ is quasilinear.
- $u_x^2 + u_y^2 = 1$ is not quasilinear.

A k-th order partial differential equation is **fully nonlinear** if the highest-order derivatives of u appear nonlinearly in the equation.

Example 7.

- $u_x^2 + u_y^2 = 1$ is fully nonlinear.
- $\text{div}\left(\dfrac{\nabla \text{u}}{\sqrt{1 + |\nabla \text{u}|^2}}\right) = 0$ is fully nonlinear.

We can also talk about systems of partial differential equations. In particular, suppose $U(\vec{x})$ is a vector-valued function on $\mathbb{R}^n$ such that $U : \mathbb{R}^n \to \mathbb{R}^m$. That is

$$U(\vec{x}) = \begin{bmatrix} u_1(\vec{x}) \\ \vdots \\ u_m(\vec{x}) \end{bmatrix}$$

where $u_1, \ldots, u_m$ are the m component functions of U. For a multi-index α, we define

$$D^\alpha U = \{D^\alpha u_1, \ldots, D^\alpha u_m\}.$$

For a nonnegative integer k, we define

$$D^k U = \{D^\alpha U : |\alpha| = k\}.$$

An equation of the form

$$\vec{F}(\vec{x}, U, DU, \ldots, D^k U) = \vec{0} \tag{1.6}$$

is a k-th order **system** of partial differential equations.

Example 8. Suppose

$$U(x,t) = \begin{bmatrix} u_1(x,t) \\ u_2(x,t) \end{bmatrix}.$$

Consider

$$\begin{cases} (u_1)_t = (u_2)_x \\ (u_2)_t = (u_1)_x. \end{cases} \tag{1.7}$$

This is a first-order system of PDEs. We note that this system can also be written as

$$U_t = AU_x \tag{1.8}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Now that we have introduced the notion of a partial differential equation and classified equations into different types, we will discuss the types of problems we are interested in studying.

1.2 Initial/Boundary Value Problems

As discussed above, a partial differential equation is an equation involving a function u and its derivatives. Given a PDE, we want to know if there are any functions u which satisfy the equation. As you may recall from studying ordinary differential equations, however, there may be no solutions or there may be many solutions for a given ODE. The same is true for partial differential equations. For example, consider the transport equation,

$$u_t - u_x = 0.$$

This equation models simple fluid motion, where $u(x,t)$ is the height of the wave at time t and position x. Clearly, $u(x,t) = 0$ satisfies this equation, but so does $u(x,t) = c$ for any constant c. In fact, there are a lot of different solutions to this equation. (We will discuss this equation more later in the course.) This is called an *ill-posed* problem, because there is not a unique solution. However, if we impose an auxiliary condition, i.e. - an initial condition, we can find a unique solution. In particular, the problem

$$\begin{cases} u_t - u_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

is a *well-posed* problem assuming the initial data $\phi(x)$ is a "nice" function.

We are interested in studying so-called *well-posed* problems. Roughly, we say a problem is *well-posed* if there exists a unique solution which depends continuously on the initial or boundary data. We will discuss particular initial value problems and boundary value problems later in the course.

The reason for asking for the function u to satisfy additional conditions (initial or boundary), is not purely a mathematical invention. As we will see more throughout the course, when modelling a physical problem with a partial differential equation, it is quite natural to ask that the solution u not only satisfies the PDE, but also satisfies some additional conditions. For example, in the transport equation above, in order to ask what the height $u(x,t)$ of the wave will be at time t and position x, we need to know something else about the wave!

1.3 General Remarks

The study of partial differential equations is a huge field. With the variety of possible PDEs, it is impossible to find a method by which we can solve all equations. In this course, we will touch upon some basic techniques for certain types of equations, but will only skim the surface of this field.

The general rule of thumb is the following. Lower-order, linear equations are easier than higher-order, nonlinear equations. Consequently, we will start with first-order linear equations and work our way up.