Math 220A - Second Practice Midterm Exam Solutions - Fall 2002

1. Classify the following in terms of degree of nonlinearity:

(a) \( u_t^2 + x^2 u_{xt} = \sin(u) \) Answer: semilinear

(b) \( u_x + [u^3]_y = x^2 + y^2 \) Answer: quasilinear

(c) \( [e^u]_x + u^2 u_y = 0 \) Answer: quasilinear

(d) \( [x^3 u]_x + y^3 u = \sin(x^2 + y^2) \) Answer: linear

(e) \( [u^3]_t + e^{u_{xt}} = 0 \) Answer: fully nonlinear

2. Solve

\[
\begin{aligned}
& u_t + u_x^2 = t \\
& u(x,0) = x.
\end{aligned}
\]

Answer: Let \( F(x, t, z, p, q) = q + p^2 - t \). Our characteristic equations are given by

\[
\begin{aligned}
\frac{dx}{ds} &= F_p = 2p \\
\frac{dt}{ds} &= F_q = 1 \\
\frac{dz}{ds} &= F_p p + q F_q = 2p^2 + q \\
\frac{dp}{ds} &= -F_x - p F_z = 0 \\
\frac{dq}{ds} &= -F_t - q F_z = 1
\end{aligned}
\]

with initial conditions

\[
\begin{aligned}
x(r,0) &= \gamma_1(r) = r \\
t(r,0) &= \gamma_2(r) = 0 \\
z(r,0) &= \phi(r) = r \\
p(r,0) &= \psi_1(r) \\
q(r,0) &= \psi_2(r)
\end{aligned}
\]

where \( \psi_1, \psi_2 \) satisfy

\[
\begin{aligned}
\phi'(r) &= \psi_1 \gamma_1' + \psi_2 \gamma_2' \\
\psi_2 + \psi_1^2 &= \gamma_2 = 0.
\end{aligned}
\]

The first equation implies \( \psi_1 = 1 \). The second equation implies \( \psi_2 = -1 \). Therefore
the solutions of our characteristic equations are given as follows,

\[ \begin{align*}
x(r, s) &= 2s + r \\
t(r, s) &= s \\
z(r, s) &= \frac{s^2}{2} + s + r \\
p(r, s) &= 1 \\
q(r, s) &= s - 1.
\end{align*} \]

Therefore, our solution is given by

\[ u(x, t) = z(r(x, t), s(x, t)) = \frac{t^2}{2} + t + x - 2t, \]

or

\[ u(x, t) = \frac{t^2}{2} + x - t. \]

3. Solve

\[
\begin{align*}
\frac{du}{dt} + 3\frac{du}{dx} - 10\frac{d^2u}{dx^2} &= 0 \\
u(x, 0) &= \phi(x) \\
v_t(x, 0) &= \psi(x)
\end{align*}
\]

by reducing the hyperbolic equation to two first-order transport equations. That is, reduce to the system

\[ \begin{align*}
(\partial_t + 5\partial_x)v &= 0 \\
(\partial_t - 2\partial_x)u &= v
\end{align*} \]

with appropriate initial conditions. Then solve these first-order equations using the method of characteristics.

**Answer:** We write our equation

\[ u_{tt} + 3u_{xt} - 10u_{xx} = 0 \]

as

\[ (\partial_t + 5\partial_x)(\partial_t - 2\partial_x)u = 0. \]

Now letting

\[ v = (\partial_t - 2\partial_x)u, \]

our equation can be written as two transport equations,

\[ \begin{align*}
u_t - 2u_x &= v \\
v_t + 5v_x &= 0.
\end{align*} \]
Now using the fact that \( v = u_t - 2u_x \), we see that \( v(x, 0) = \psi(x) - 2\phi'(x) \). Therefore, we must first solve the initial-value problem

\[
\begin{align*}
v_t + 5v_x &= 0 \\
v(x, 0) &= \psi(x) - 2\phi'(x).
\end{align*}
\]

We know solutions of the transport equation are given by

\( v(x, t) = f(x - 5t) \).

Combining this with our initial condition, we have

\( v(x, t) = \psi(x - 5t) - 2\phi'(x - 5t) \).

Next, we solve

\[
\begin{align*}
 u_t - 2u_x &= \psi(x - 5t) - 2\phi'(x - 5t) \\
 u(x, 0) &= \phi(x)
\end{align*}
\]

Our characteristic ODE are given by

\[
\begin{align*}
 \frac{dx}{ds} &= -2 \\
 \frac{dt}{ds} &= 1 \\
 \frac{dz}{ds} &= \psi(x(s) - 5t(s)) - 2\phi'(x(s) - 5t(s))
\end{align*}
\]

subject to the initial conditions

\[
\begin{align*}
x(r, 0) &= r \\
t(r, 0) &= 0 \\
z(r, 0) &= \phi(r).
\end{align*}
\]

The solutions of this system of ODEs is given by

\[
\begin{align*}
x(r, s) &= -2s + r \\
t(r, s) &= s \\
z(r, s) &= \int_0^s \psi(-2s' + r - 5s')\,ds' - 2 \int_0^2 \phi'(-2s' + r - 5s')\,ds' + \phi(r) \\
&= \int_0^s \left[ \psi(-7s' + r) - 2\phi'(-7s' + r) \right]\,ds' + \phi(r) \\
&= -\frac{1}{7} \int_r^{-7s+r} [\psi(y) - 2\phi'(y)]\,dy + \phi(r).
\end{align*}
\]
Therefore, our solution \( u \) is given by

\[
\begin{align*}
    u(x, t) = z(r(x, t), s(x, t)) &= -\frac{1}{t} \int_{x+2t}^{x-5t} [\psi(y) - 2\phi'(y)] dy + \phi(x + 2t)
\end{align*}
\]

which implies

\[
\begin{align*}
    u(x, t) &= \frac{5}{t}\phi(x + 2t) + \frac{2}{t}\phi(x - 5t) + \frac{1}{t} \int_{x-5t}^{x+2t} \psi(y) dy.
\end{align*}
\]

4. Find the unique, weak solution of the following which satisfies the entropy condition,

\[
\begin{align*}
    \begin{cases}
        u_t - (\sin(u))_x = 0 & t \geq 0 \\
        u(x, 0) = \phi(x)
    \end{cases}
\end{align*}
\]

in each of the two cases below.

**Answer:** First, our characteristic ODEs are given by

\[
\begin{align*}
    \frac{dt}{ds} &= 0 \\
    \frac{dx}{ds} &= -\cos(z) \\
    \frac{dz}{ds} &= 0
\end{align*}
\]

with initial conditions

\[
\begin{align*}
    t(r, 0) &= 0 \\
    x(r, 0) &= r \\
    z(r, 0) &= \phi(r).
\end{align*}
\]

Therefore, our projected characteristics are given by

\[
x = -\cos(\phi(r))t + r.
\]

(a)

\[
\phi(x) = \begin{cases}
    0 & x < 0 \\
    \pi & x > 0.
\end{cases}
\]

In this case, the characteristics are given by

\[
\begin{align*}
    x &= -t + r & r < 0 \\
    x &= t + r & r > 0.
\end{align*}
\]

Therefore, we need to fill in the open wedge with a rarefaction wave,

\[
G(x/t) = (f')^{-1}(x/t) = \arccos(-x/t).
\]
Therefore, our solution is given by

\[
\begin{cases}
0 & x < -t \\
\arccos(-x/t) & -t < x < t \\
\pi & x > t.
\end{cases}
\]

(b)

\[
\phi(x) = \begin{cases}
\pi & x < 0 \\
0 & x > 0.
\end{cases}
\]

In this case, the characteristics are given by

\[
x = t + r \quad r < 0 \\
x = -t + r \quad r > 0.
\]

Therefore, our projected characteristics intersect. We introduce a shock curve. This curve \( x = \xi(t) \) must satisfy

\[
\xi'(t) = \frac{[f(u)]}{[u]} = \frac{-\sin(\pi) + \sin(0)}{\pi - 0} = 0,
\]

and pass through the origin. Therefore, the curve is \( x = 0 \). Therefore, our solution is given by

\[
\begin{cases}
\pi & x < 0 \\
0 & x > 0.
\end{cases}
\]

5. We say \( u \) is a weak solution of

\[
(*) \begin{cases}
[g(u)]_t + [f(u)]_x = 0 \\
u(x, 0) = \phi(x)
\end{cases}
\]

if \( u \) satisfies

\[
\int_0^\infty \int_{-\infty}^{\infty} g(u)v_t + f(u)v_x \, dx \, dt + \int_{-\infty}^{\infty} \phi(x)v(x, 0) \, dx = 0
\]

for all \( v \in C^\infty(\mathbb{R} \times [0, \infty)) \) with compact support. Suppose \( u \) is a weak solution of (*) such that \( u \) has a jump discontinuity across the curve \( x = \xi(t) \), but \( u \) is smooth on either side of the curve \( x = \xi(t) \). Let \( u^-(x, t) \) be the value of \( u \) to the left of the curve and \( u^+(x, t) \) be the value of \( u \) to the right of the curve. Prove that \( u \) must satisfy the condition

\[
\frac{[f(u)]}{[g(u)]} = \xi'(t)
\]

across the curve of discontinuity, where

\[
[f(u)] = f(u^-) - f(u^+) \quad [g(u)] = g(u^-) - g(u^+).
\]
**Answer:** If \( u \) is a weak solution of (*) then

\[
\int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx = 0
\]

for all smooth functions \( v \in C^\infty(\mathbb{R} \times [0, \infty)) \) with compact support. Let \( v \) be a smooth function such that \( v(x,0) = 0 \), and break up the first integral into the regions \( \Omega^- \), \( \Omega^+ \) where

\[
\Omega^- \equiv \{(x,t) : 0 < t < \infty, -\infty < x < \xi(t)\}
\]

\[
\Omega^+ \equiv \{(x,t) : 0 < t < \infty, \xi(t) < x < +\infty\}.
\]

Therefore,

\[
0 = \int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx
\]

\[
= \iint_{\Omega^-} [g(u)v_t + f(u)v_x] \, dx \, dt + \iint_{\Omega^+} [g(u)v_t + f(u)v_x] \, dx \, dt.
\]

Combining the Divergence Theorem with the fact that \( v \) has compact support and \( v(x,0) = 0 \), we have

\[
\iint_{\Omega^-} [g(u)v_t + f(u)v_x] \, dx \, dt = -\iint_{\Omega^-} [(g(u))_t + (f(u))_x]v \, dx \, dt
\]

\[
+ \int_{x=\xi(t)} [g(u^-)v_2 + f(u^-)v_1] \, ds
\]

where \( \nu = (\nu_1, \nu_2) \) is the outward unit normal to \( \Omega^- \).

Similarly, we see that

\[
\iint_{\Omega^+} [g(u)v_t + f(u)v_x] \, dx \, dt = -\iint_{\Omega^+} [(g(u))_t + (f(u))_x]v \, dx \, dt
\]

\[
- \int_{x=\xi(t)} [g(u^+)v_2 + f(u^+)v_1] \, ds.
\]

By assumption, \( u \) is a weak solution of

\[
[g(u)]_t + [f(u)]_x = 0
\]
and $u$ is smooth on either side of $x = \xi(t)$. Therefore, $u$ is a strong solution on either side of the curve of discontinuity. Consequently, we see that

$$\int_{\Omega^-} [(g(u))_t + (f(u))_x]v \, dx \, dt = 0 = \int_{\Omega^+} [(g(u))_t + (f(u))_x]v \, dx \, dt.$$  

Combining these facts, we see that

$$\int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1] \, ds - \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1] \, ds = 0.$$

Since this is true for all smooth functions $v$, we have

$$g(u^-)\nu_2 + f(u^-)\nu_1 = g(u^+)\nu_2 + f(u^+)\nu_1,$$

which implies

$$\frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = -\frac{\nu_2}{\nu_1}.$$

Now the curve $x = \xi(t)$ has slope given by the negative reciprocal of the normal to the curve; that is,

$$\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2}.$$

Therefore,

$$\xi'(t) = -\frac{\nu_2}{\nu_1} = \frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = \frac{[f(u)]}{[g(u)]},$$

as claimed.