1. Classify the following in terms of degree of nonlinearity:

(a) \( u_t + e^u u_x = x^2 \) **Answer:** quasilinear

(b) \( u_t + x^2 u_x = e^u \) **Answer:** semilinear

(c) \( x^3 u_t + u_x^2 = 1 \) **Answer:** fully nonlinear

(d) \( u_t + x^3 u_x = \sin(x^2) \) **Answer:** linear

(e) \( u_{xt} + \left[ \frac{u_x^2}{2} \right]_x = \cos(u_x) \) **Answer:** semilinear

2. Find the unique weak solution of

\[
\begin{align*}
  u_t + uu_x &= 0 \\
  u(x,0) &= \phi(x)
\end{align*}
\]

where

\[
\phi(x) = \begin{cases} 
  2 & x < 0 \\
  1 & 0 < x < 1 \\
  0 & x > 1 
\end{cases}
\]

which satisfies the entropy condition.

**Answer:** The characteristic ODE are given by

\[
\begin{align*}
  \frac{dt}{ds} &= 1 & t(r,0) &= 0 \\
  \frac{dx}{ds} &= z & x(r,0) &= r \\
  \frac{dz}{ds} &= 0 & z(r,0) &= \phi(r)
\end{align*}
\]

which implies the projected characteristics are given by \( x = \phi(r)t + r \). For \( r < 0 \), our projected characteristics are given by \( x = 2t + r \). For \( 0 < r < 1 \), our projected characteristics are given by \( x = t + r \). For \( r > 1 \), our projected characteristics are given by \( x = r \).
Therefore, we need to put in two shock curves.

First, between $u^- = 2$ and $u^+ = 1$, our shock curve $x = \xi_1(t)$ needs to satisfy

$$\xi'_1(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{2^2}{2} - \frac{1^2}{2}}{2 - 1} = \frac{3}{2}.$$ 

In addition, this curve needs to pass through the point $(0, 0)$. Therefore, $\xi_1(t) = \frac{3}{2} t$.

Second, between $u^- = 1$ and $u^+ = 0$, we need the shock curve $x = \xi_2(t)$ to satisfy

$$\xi'_2(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{1^2}{2} - \frac{0^2}{2}}{1 - 0} = \frac{1}{2}.$$ 

In addition, this curve needs to pass through the point $(1, 0)$. Therefore, $\xi_2(t) = \frac{1}{2} t + 1$.

Therefore, our solution is given by

$$u(x, t) = \begin{cases} 
2 & x < \frac{3}{2} t \\
1 & \frac{3}{2} t < x < \frac{1}{2} t + 1 \\
0 & x > \frac{1}{2} t + 1.
\end{cases}$$

However, at $t = 1$, the two shock curves intersect. Therefore, the above solution is only valid for $0 \leq t \leq 1$. Consequently, we need to define a new shock curve $x = \xi_3(t)$
starting from the point \((3/2, 1)\) which satisfies the Rankine-Hugoniot jump condition where \(u^- = 2\) and \(u^+ = 0\). That is,

\[
\xi_3'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{2^2}{2} - 0^2}{2 - 0} = 1.
\]

Therefore, \(\xi_3(t) = t + \frac{1}{2}\).

In summary, for \(0 \leq t \leq 1\) our solution is given by

\[
\begin{array}{l}
\begin{cases}
2 & x < \frac{3}{2} t \\
1 & \frac{3}{2} t < x < \frac{1}{2} t + 1 \\
0 & x > \frac{1}{2} t + 1.
\end{cases}
\end{array}
\]

While for \(t \geq 1\) our solution is given by

\[
\begin{array}{l}
\begin{cases}
2 & x < t + \frac{1}{2} \\
0 & x > t + \frac{1}{2}
\end{cases}
\end{array}
\]

3. Solve

\[
\begin{cases}
u_t + u_x^2 = 0 \\
u(x, 0) = -x^2
\end{cases}
\]

Find the time \(T\) for which \(|u| \to +\infty\) as \(t \to T\).

Answer: Writing our equation as \(F(x, t, z, p, q) = p^2 + q\), our characteristic ODE are
given by
\[
\begin{align*}
\frac{dx}{ds} &= 2p & x(r, 0) &= r \\
\frac{dt}{ds} &= 1 & t(r, 0) &= 0 \\
\frac{dz}{ds} &= p^2 & z(r, 0) &= -r^2 \\
\frac{dp}{ds} &= 0 & p(r, 0) &= -2r \\
\frac{dq}{ds} &= 0 & q(r, 0) &= -4r^2.
\end{align*}
\]

Note the initial conditions for \( p \) and \( q \), denoted \( p(r, 0) = \psi_1(r) \) and \( q(r, 0) = \psi_2(r) \) come from the two equations
\[
\psi_1^2 + \psi_2 = 0 \\
\phi' = \psi_1 \gamma_1' + \psi_2 \gamma_2'.
\]

Now solving this system of ODEs, we have
\[
\begin{align*}
x &= -4rs + r \\
t &= s \\
z &= 4r^2s - r^2 \\
p &= -2r \\
q &= -4r^2.
\end{align*}
\]

Solving for \( r \) and \( s \) in terms of \( x \) and \( t \), we have
\[
\begin{aligned}
u(x, t) &= 4 \left( \frac{x}{1 - 4t} \right)^2 t - \left( \frac{x}{1 - 4t} \right)^2.
\end{aligned}
\]

The solution is finite up until \( T = 1/4 \).

4. Solve
\[
\begin{align*}
u_x + xu_y - 4u &= 0 \\
u(1, y) &= y^2.
\end{align*}
\]

Answer: Our characteristic ODE are given by
\[
\begin{align*}
\frac{dx}{ds} &= 1 & x(r, 0) &= 1 \\
\frac{dy}{ds} &= x & y(r, 0) &= r \\
\frac{dz}{ds} &= 4z & z(r, 0) &= r^2.
\end{align*}
\]
Solving the characteristic ODE, we have
\[
\begin{align*}
  x &= s + 1 \\
  y &= s^2/2 + s + r \\
  z &= r^2 e^{4s}.
\end{align*}
\]

Solving for \( r \) and \( s \) in terms of \( x \) and \( y \), we see our solution is given by
\[
 u(x, y) = \left[ y - \frac{(x - 1)^2}{2} - (x - 1) \right] e^{4(x-1)}.
\]

5. Find the unique weak solution of
\[
\begin{align*}
  &u_t + u^2 u_x = 0 \\
  &u(x, 0) = \phi(x)
\end{align*}
\]
where
\[
\phi(x) = \begin{cases} 
  0 & x < 0 \\
  1 & 0 < x < 2 \\
  0 & x > 2
\end{cases}
\]
which satisfies the Oleinik entropy condition.

**Answer:** As in problem 2, the projected characteristics are given by \( x = \phi(r)t + r \). For \( r < 0 \), \( \phi(r) = 0 \) implies the proj. chars. are given by \( x = r \) for \( r < 0 \). For \( 0 < r < 2 \), \( \phi(r) = 1 \) implies the proj. chars. are given by \( x = t + r \) for \( 0 < r < 2 \). For \( r > 1 \), \( \phi(r) = 1 \) implies the proj. chars. are given by \( x = r \) for \( r > 2 \).

We put in a rarefaction wave in the open wedge and introduce a shock curve where the projected characteristics cross. The rarefaction wave is given by \( G(x/t) = (f')^{-1}(x/t) = \sqrt{x/t} \). The shock curve \( x = \xi_1(t) \) must satisfy the R-H jump condition,
\[
\xi'_1(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{1^3 - 0^3}{1 - 0} = \frac{1}{3}.
\]

Notice that the Oleinik entropy condition is satisfied because the chord connecting \((u^-, f(u^-)) = (1, 1/3)\) and \((u^+, f(u^+)) = (0, 0)\) lies above the graph of the function \( f \).
Combining the rarefaction wave and shock curve defined above, we see our solution is given by

\[
  u(x, t) = \begin{cases} 
  0 & x < 0 \\
  \sqrt{\frac{x}{t}} & 0 < x < t \\
  1 & t < x < \frac{1}{3}t + 2 \\
  0 & x > \frac{1}{3}t + 2 
  \end{cases}
\]

Now at \( t = 3 \), the rarefaction wave hits the shock curve. Therefore, we need to introduce a new shock curve. The new shock curve \( x = \xi'_2(t) \) must satisfy

\[
  \xi'_2(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{\frac{1}{3} \left( \sqrt{\frac{x}{t}} \right)^3 - \frac{9x^3}{3}}{\sqrt{\frac{x}{t}} - 0} = \frac{x}{3t}.
\]

In addition, this curve must pass through the point \((x, t) = (3, 3)\). Therefore, this curve is given by \( x = (9t)^{1/3} \). Again, notice that the Oleinik entropy condition is satisfied along this shock curve.

In summary, our solution is given as follows. For \( 0 \leq t < 3 \),

\[
  u(x, t) = \begin{cases} 
  0 & x < 0 \\
  \sqrt{\frac{x}{t}} & 0 < x < t \\
  1 & t < x < \frac{1}{3}t + 2 \\
  0 & x > \frac{1}{3}t + 2 
  \end{cases}
\]

While for \( t \geq 3 \), our solution is given by

\[
  u(x, t) = \begin{cases} 
  0 & x < 0 \\
  \sqrt{\frac{x}{t}} & 0 < x < (9t)^{1/3} \\
  0 & x > (9t)^{1/3}.
  \end{cases}
\]
6. Let \( f, g \) be \( C^\infty \) functions. Consider the initial value problem

\[
\begin{aligned}
\left\{ \begin{array}{l}
[g(u)]_t + [f(u)]_x = 0 \\
u(x, 0) = \phi(x)
\end{array} \right.
\end{aligned}
\]

(a) Give a definition for a strong solution of this initial value problem.

**Answer:** We say \( u \) is a strong solution of the initial-value problem above if \( u \) is a continuously differentiable function which satisfies the equation \( [g(u)]_t + [f(u)]_x = 0 \) at each point \((x, t)\) and \( u(x, 0) = \phi(x) \).

(b) Give a definition for a weak solution of this initial value problem.

**Answer:** We say \( u \) is a weak solution of the initial-value problem above if \( u \) satisfies

\[
\int_0^\infty \int_{-\infty}^{\infty} [g(u)v_t + f(u)v_x ] \, dx \, dt + \int_{-\infty}^{\infty} g(\phi(x))v(x, 0) \, dx = 0
\]

for all smooth functions \( v \) with compact support.

(c) Prove that any strong solution is in fact a weak solution.

**Answer:** If \( u \) is strong, then

\[
[g(u)]_t + [f(u)]_x = 0.
\]

Now multiplying this equation by a smooth function \( v \) with compact support and integrating over \( \mathbb{R} \times [0, \infty) \), we have

\[
\int_0^\infty \int_{-\infty}^{\infty} \{ [g(u)]_t + [f(u)]_x \} \, v \, dx \, dt = 0.
\]
Integrating by parts, we have
\[
- \int_0^\infty \int_{-\infty}^\infty g(u) v_t \, dx \, dt + \int_{-\infty}^\infty g(u) v \bigg|_{t=0}^{t=\infty} - \int_0^\infty \int_{-\infty}^\infty f(u) v_x \, dx \, dt + \int_0^\infty f(u) v \bigg|_{x=-\infty}^{x=\infty} = 0.
\]
Using the fact that \( v \) has compact support, three of the boundary terms vanish. Therefore, we are left with
\[
- \int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt - \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx = 0,
\]
as desired.

7. (a) Find the general solution of
\[
u_{tt} + 2u_{xt} - 3u_{xx} = 0.
\]

**Answer:** We rewrite our equation as
\[
(\partial_t + 3\partial_x)(\partial_t - \partial_x)u = 0.
\]
Then we introduce new variables \( \xi, \eta \) such that
\[
\begin{align*}
\partial_\xi &= \partial_t + 3\partial_x \\
\partial_\eta &= \partial_t - \partial_x.
\end{align*}
\]
Therefore,
\[
\begin{align*}
\xi &= \frac{1}{4}(x + t) \\
\eta &= -\frac{1}{4}(x - 3t).
\end{align*}
\]
With this change of variables, we have
\[
u_{\xi\eta} = 0.
\]
Therefore, the general solution is given by
\[
u(x,t) = f(x + t) + g(x - 3t).
\]

(b) Find the solution of the initial-value problem,
\[
\begin{cases}
u_{tt} + 2u_{xt} - 3u_{xx} = 0 \\
u(x,0) = \phi(x) \\
u_t(x,0) = \psi(x).
\end{cases}
\]
**Answer:** Our general solution is given by

\[ u(x, t) = f(x + t) + g(x - 3t). \]

We want

\[
\begin{align*}
  u(x, 0) &= f(x) + g(x) = \phi(x) \\
  u_t(x, 0) &= f'(x) - 3g'(x) = \psi(x).
\end{align*}
\]

Solving these equations for \( f \) and \( g \), we arrive at the following solution for our initial-value problem,

\[
\begin{align*}
  u(x, t) &= \frac{3}{4} \phi(x + t) + \frac{1}{4} \phi(x - 3t) + \frac{1}{4} \int_{x-3t}^{x+t} \psi(y) \, dy.
\end{align*}
\]