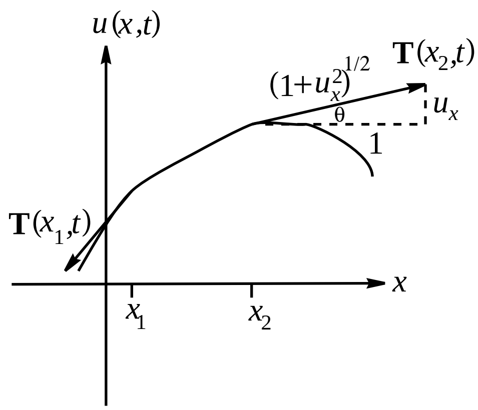
5 Wave Equation in $\mathbb{R}$

5.1 Derivation

Ref: Strauss, Section 1.3, Evans, Section 2.4

Consider a homogeneous string of length l and density $\rho = \rho(x)$. Assume the string is moving in the transverse direction, but not in the longitudinal direction. Let $u(x,t)$ denote the displacement of the string from equilibrium at time t and position x. Therefore, the slope of the string at time t, position x is given by $u_x(x,t)$. Let $T(x,t)$ be the magnitude of the tension (force) tangential to the string at time t position x.



Consider the part of the string between the points x_1 and x_2. The net force acting on the string in the longitudinal direction (x), denoted F_1, between the points x_1 and x_2 is given by

$$\begin{aligned}
F_1\big|_{x_1}^{x_2} &= T(x,t)\cos\theta\big|_{x_1}^{x_2} \\
&= T(x,t)\left.\frac{1}{\sqrt{1+u_x^2}}\right|_{x_1}^{x_2} \\
&= T(x_2,t)\frac{1}{\sqrt{1+u_x^2(x_2,t)}} - T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}}.
\end{aligned}$$

But, by assumption, the string is not moving in the longitudinal direction, and, therefore, the acceleration in the longitudinal direction is zero. Consequently, using Newton's law, $\mathbf{F} = m\mathbf{a}$, we conclude that

$$F_1(x,t)\big|_{x_1}^{x_2} = T(x_2,t)\frac{1}{\sqrt{1+u_x^2(x_2,t)}} - T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} = 0. \tag{5.1}$$

In the transverse direction, the force acting on the string between the points x_1 and x_2

at time t, denoted F_2, is given by

$$\begin{aligned}
F_2\big|_{x_1}^{x_2} &= T(x,t)\sin\theta\big|_{x_1}^{x_2} \\
&= T(x,t)\,\frac{u_x}{\sqrt{1+u_x^2}}\Bigg|_{x_1}^{x_2} \\
&= T(x_2,t)\frac{u_x(x_2,t)}{\sqrt{1+u_x^2(x_2,t)}} - T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}}.
\end{aligned}$$

Again, we are assuming that all motion of the string is purely in the transverse direction. By Newton's law, $\mathbf{F} = m\mathbf{a}$ implies $F_2(x,t) = ma_2(x,t)$ where $a_2(x,t)$ denotes the component of the acceleration of the string in the transverse direction at position x, time t. Therefore, between the points x_1 and x_2,

$$F_2(x,t)\big|_{x_1}^{x_2} = \int_{x_1}^{x_2} \rho u_{tt}(x,t)\,dx.$$

Therefore, we have

$$T(x_2,t)\frac{u_x(x_2,t)}{\sqrt{1+u_x^2(x_2,t)}} - T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} = \int_{x_1}^{x_2} \rho u_{tt}(x,t)\,dx. \tag{5.2}$$

Now if we assume u_x is small (meaning small vibrations of the string), then

$$\sqrt{1+u_x^2} \approx 1.$$

This can be justified by the Taylor series expansion,

$$\sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \cdots$$

Therefore for u_x small, by (5.2), we have

$$T(x_2,t)u_x(x_2,t) - T(x_1,t)u_x(x_1,t) \approx \int_{x_1}^{x_2} \rho u_{tt}(x,t)\,dx.$$

Multiplying this equation by $\frac{1}{x_2-x_1}$ and taking the limit as $x_2 \to x_1$, we have

$$\begin{aligned}
\lim_{x_2\to x_1}\frac{1}{x_2-x_1}(T(x_2,t)u_x(x_2,t) - T(x_1,t)u_x(x_1,t)) &\approx \lim_{x_2\to x_1}\frac{1}{x_2-x_1}\int_{x_1}^{x_2} \rho u_{tt}(x,t)\,dx \\
\implies (T(x,t)u_x(x,t))_x &\approx \rho u_{tt}.
\end{aligned}$$

Therefore, the equation

$$(Tu_x)_x = \rho u_{tt} \tag{5.3}$$

gives us a simplified model for the motion of the string.

By assuming u_x is small, by (5.1), we have

$$T(x_2,t) \approx T(x_1,t)$$

which means T is independent of x, and, therefore, the tension is constant along the string. If we also assume T is independent of t and ρ is constant along the string, equation (5.3) can be simplified to

$$u_{tt} = \frac{T}{\rho} u_{xx}.$$

In general, T and ρ are nonnegative. Therefore, letting

$$c = \sqrt{\frac{T}{p}}$$

our equation becomes

$$u_{tt} = c^2 u_{xx}.$$

This is known as the *wave equation*. We will see later that c represents the *wave speed*.

5.2 General Solution for Wave Equation in $\mathbb{R}$

Ref: Strauss, Section 2.1

Claim 1. *The general solution of*

$$u_{tt} - c^2 u_{xx} = 0 \qquad x \in \mathbb{R} \tag{5.4}$$

is given by

$$u(x,t) = f(x+ct) + g(x-ct) \tag{5.5}$$

for (smooth) functions f and g.

Proof of Claim 1. (Method 1: Reduction to First-Order Equations)
Consider

$$u_{tt} = c^2 u_{xx} \qquad x \in \mathbb{R}$$

This equation can be rewritten as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0.$$

Let

$$v \equiv u_t - cu_x.$$

Therefore,

$$(\partial_t + c\partial_x)v = 0.$$

That is, v is solves a first-order transport equation. Consequently, the general solution for v is given by

$$v = h(x - ct).$$

Now it remains to solve

$$u_t - cu_x = h(x - ct). \tag{5.6}$$

Using the method of characteristics, we define the characteristic equations as

$$\frac{dt}{ds} = 1$$
$$\frac{dx}{ds} = -c$$
$$\frac{dz}{ds} = h(x - ct).$$

One solution of this system is $t = s$, $x = -cs$ and $dz/ds = h(-2cs)$ which implies

$$z(s) = -\frac{1}{2c}\int_0^{-2cs} h(\tau)\,d\tau.$$

Letting $u(x(s), t(s)) = z(s)$, we arrive at a particular solution of (5.6) of the form

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\int_{x-ct}^{0} h(s)\,ds \\ &\equiv g(x - ct). \end{aligned}$$

We also note that any function of the form $u(x,t) = f(x + ct)$ satisfies the homogeneous equation

$$u_t - cu_x = 0.$$

Therefore, any function of the form

$$u(x,t) = f(x + ct) + g(x - ct)$$

will give us a solution of (5.4).

To show that we have found *all* of the solutions of (5.4), we introduce the function w by defining $w = w(x,t)$ as follows. Let u be a solution of the wave equation. Therefore,

$$u_t - cu_x = h(x - ct)$$

for some function h. Now let $\widetilde{f}$ be an arbitrary smooth function and define

$$w(x,t) = u(x,t) + \widetilde{f}(x + ct) - \frac{1}{2c}\int_{x-ct}^{0} h(s)\,ds.$$

It is straightforward to show that

$$w_t - cw_x = 0,$$

and, therefore, $w(x,t) = k(x + ct)$ for some function k. Consequently,

$$u(x,t) = f(x + ct) + g(x - ct)$$

where $f(x + ct) \equiv k(x + ct) - \widetilde{f}(x + ct)$ and $g(x - ct) = \frac{1}{2c}\int_{x-ct}^{0} h(s)\,ds$. But, $k, \widetilde{f}, h$ were arbitrary. Therefore, the general solution of (5.4) is given by (5.5). □

Proof of Claim 1. (Method 2: Characteristic Coordinates)

Another way of deriving the general solution (5.5) of (5.4) is by making a change of variables. Rewriting

$$u_{tt} - c^2 u_{xx} = 0.$$

as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0,$$

we would like to introduce coordinates ξ, η such that

$$\begin{aligned} \partial_\xi &= \partial_t + c\partial_x \\ \partial_\eta &= \partial_t - c\partial_x. \end{aligned}$$

That is, we want

$$\begin{aligned} t_\xi &= 1 \qquad & t_\eta &= 1 \\ x_\xi &= c \qquad & x_\eta &= -c. \end{aligned}$$

That is,

$$\begin{aligned} t &= \xi + \eta \\ x &= c\xi - c\eta, \end{aligned}$$

which implies

$$\begin{aligned} \xi &= \frac{1}{2c}(x + ct) \\ \eta &= -\frac{1}{2c}(x - ct) \end{aligned}$$

For simplicity, we make a change of scale, introducing the *characteristic coordinates*

$$\begin{aligned} \widetilde{\xi} &= 2c\xi = x + ct \\ \widetilde{\eta} &= -2c\eta = x - ct. \end{aligned}$$

In these new coordinates, we have

$$\begin{aligned} \partial_{\widetilde{\xi}} &= \frac{1}{2c}\partial_\xi = \frac{1}{2c}[\partial_t + c\partial_x] \\ \partial_{\widetilde{\eta}} &= -\frac{1}{2c}\partial_\eta = -\frac{1}{2c}[\partial_t - c\partial_x] \end{aligned}$$

which implies

$$-4c^2 \partial_{\widetilde{\xi}} \partial_{\widetilde{\eta}} u = (\partial_t + c\partial_x)(\partial_t - c\partial_x)u = 0.$$

Consequently,

$$u_{\widetilde{\xi}\widetilde{\eta}} = 0,$$

which implies

$$\begin{aligned} u &= f(\widetilde{\xi}) + g(\widetilde{\eta}) \\ &= f(x + ct) + g(x - ct) \end{aligned}$$

for arbitrary functions f and g, thus proving our claim. $\square$

Geometric Interpretation. The general solution of the wave equation is the sum of two arbitrary functions f and g where $f = f(x+ct)$ and $g = g(x-ct)$. In particular, $f(x+ct)$ is a wave moving to the left with speed c, while $g(x-ct)$ is a wave moving to the right with speed c.

5.3 Initial Value Problem

Consider the following initial-value problem:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{5.7}$$

As should be familiar from ODE theory, we need to prescribe two pieces of initial data to hope to get a unique solution. In the previous section, we showed that

$$u(x,t) = f(x+ct) + g(x-ct)$$

is the general solution of the PDE. We look for a solution of this form which will satisfy our initial data. This means we need

$$\begin{aligned} u(x,0) &= f(x) + g(x) = \phi(x) \\ u_t(x,0) &= cf'(x) - cg'(x) = \psi(x). \end{aligned}$$

Solving this system, we get

$$\begin{aligned} f' &= \frac{1}{2}\left(\phi' + \frac{\psi}{c}\right) \\ g' &= \frac{1}{2}\left(\phi' - \frac{\psi}{c}\right) \end{aligned}$$

which implies

$$\begin{aligned} f(s) &= \frac{1}{2}\phi(s) + \frac{1}{2c}\int_0^s \psi + C_1 \\ g(s) &= \frac{1}{2}\phi(s) - \frac{1}{2c}\int_0^s \psi + C_2. \end{aligned}$$

Using the fact that

$$\phi(x) = f(x) + g(x),$$

we see that $C_1 + C_2 = 0$. Therefore, we conclude that

$$\begin{aligned} u(x,t) &= f(x+ct) + g(x-ct) \\ &= \left[\frac{1}{2}\phi(x+ct) + \frac{1}{2c}\int_0^{x+ct} \psi\right] + \left[\frac{1}{2}\phi(x-ct) - \frac{1}{2c}\int_0^{x-ct} \psi\right], \end{aligned}$$

which simplifies to

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds.} \tag{5.8}$$

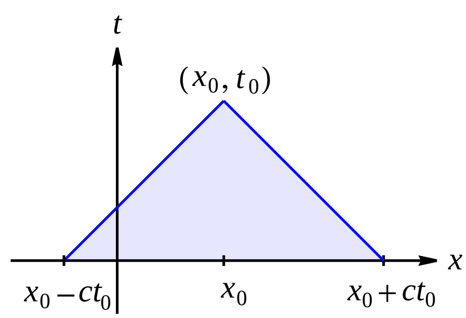
This solution formula (5.8) is known as **d'Alembert's formula** for the *unique* solution of the initial-value problem (5.7) for the wave equation on $\mathbb{R}$.

5.4 Energy Methods

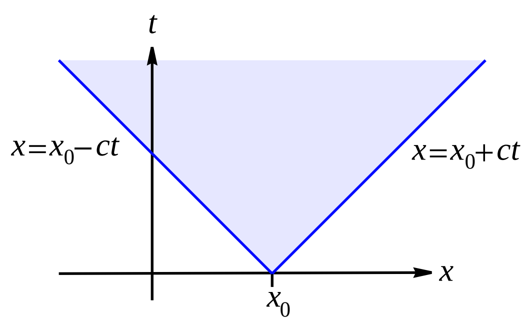
Ref: Strauss, Section 2.2; Evans, Section 2.4.3

5.4.1 Domain of Dependence

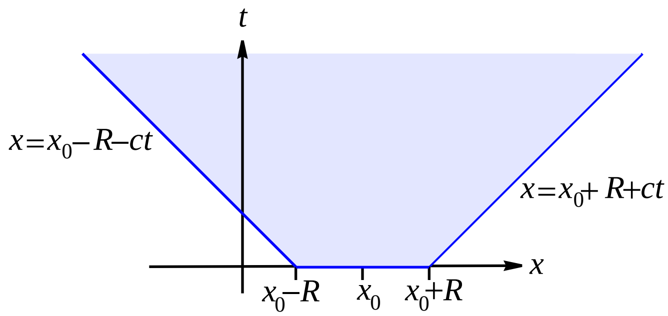
By d'Alembert's formula (5.8) for solutions of the wave equation, we see that the value of u at any point $(x_0, t_0) \in \mathbb{R}^2$ depends only on the values of the initial data in the interval $[x_0 - ct_0, x_0 + ct_0]$. That is to say, the **domain of dependence** for the point (x_0, t_0) is the cone $\{(x,t) : x_0 - c(t_0 - t) \leq x \leq x_0 + c(t_0 - t)\}$.



Similarly, we see that the initial condition at the point $(x_0, 0)$ affects only that part of the solution in the cone $\{(x,t) : t \geq 0, x_0 - ct \leq x \leq x_0 + ct\}$. This region is known as the **domain of influence** of the point $(x_0, 0)$.



Therefore, if the initial data is supported in an interval $\{x : |x - x_0| \leq R\}$, then the solution u is supported in the region $\{(x,t) : t \geq 0, x_0 - R - ct \leq x \leq x_0 + R + ct\}$.



Therefore, for initial data with compact support, the solution $u(x,t)$ will have compact support in $\mathbb{R}$ for any time t. This phenomenon is known as the **finite propagation speed** for the wave equation.

5.4.2 Energy

We now define an *energy* associated with solutions of the wave equation. In general, an energy associated with a PDE is a quantity which is conserved for a solution u over time. For a solution u of the wave equation on $\mathbb{R}$, we define the **energy** of u at time t as

$$E(t) \equiv \frac{1}{2}\int_{-\infty}^{\infty} [u_t^2(x,t) + c^2 u_x^2(x,t)]\,dx. \tag{5.9}$$

Claim 2. *Let u be a solution of the initial-value problem* (5.7)*. Assume the initial data ϕ and ψ have compact support. Then the energy* (5.9) *of the solution u is a conserved quantity. That is, $E'(t) = 0$, and, therefore, $E(t) = E(0)$.*

Proof. From the definition of energy (5.9), we have

$$E'(t) = \frac{1}{2}\int_{-\infty}^{\infty} [2u_t u_{tt} + 2c^2 u_x u_{xt}]\,dx.$$

Integrating by parts, we see that

$$\int_{-\infty}^{\infty} c^2 u_x u_{xt}\,dx = \lim_{b\to+\infty} c^2 u_x u_t\big|_{x=-b}^{x=b} - \int_{-\infty}^{\infty} c^2 u_{xx} u_t\,dx.$$

Combining the fact that ϕ and ψ have compact support with the finite propagation speed described in the previous section, we know that for any time $t > 0$, $u(x,t)$ has compact support. Therefore, the boundary terms above drop out. Consequently, we have

$$\begin{aligned} E'(t) &= \frac{1}{2}\int_{-\infty}^{\infty} [2u_t u_{tt} - 2c^2 u_{xx} u_t]\,dx \\ &= \int_{-\infty}^{\infty} u_t[u_{tt} - c^2 u_{xx}]\,dx. \end{aligned}$$

But, u is a solution of the wave equation. Therefore, $u_{tt} - c^2 u_{xx} = 0$. Consequently, we have shown that $E'(t) = 0$, as desired.

□

One question you may ask is how the energy (5.9) above was defined. How did we know this was going to be a conserved quantity? There is a relation between the energy defined above and the traditional notion of kinetic energy. In particular, from physics, the kinetic energy of a moving body is defined to be $\frac{1}{2}mv^2$. Therefore, for a wave u, we can define its kinetic energy as

$$KE = \frac{1}{2}\int \rho u_t^2 \, dx,$$

which is a part of the energy defined above (multiplied by an extra constant factor ρ). Then, we can think of the potential energy as

$$PE = \frac{1}{2}\int \rho u_x^2.$$

Alternatively, however, we can think of the energy of a moving body as any conserved quantity. With this idea, let's see if we can find a conserved quantity for a solution u of the wave equation in $\mathbb{R}$.

Suppose u is a solution of the wave equation in $\mathbb{R}$ such that u has compact support. Therefore, u satisfies

$$u_{tt} - c^2 u_{xx} = 0.$$

Now multiply this equation by u_t and integrate over $\mathbb{R}$. Doing so, we have

$$0 = \int_{-\infty}^{\infty} u_t[u_{tt} - c^2 u_{xx}]\, dx = \int_{-\infty}^{\infty} \left(\frac{1}{2}[(u_t)^2]_t - c^2 u_t u_{xx}\right) dx. \tag{5.10}$$

Now, as we did above, we can integrate this second term by parts,

$$\begin{aligned} -c^2 \int_{-\infty}^{\infty} u_t u_{xx}\, dx &= c^2 \int_{-\infty}^{\infty} u_{xt} u_x \, dx \\ &= c^2 \int_{-\infty}^{\infty} \left(\frac{1}{2}[(u_x)^2]_t\right) dx, \end{aligned} \tag{5.11}$$

using the fact that the boundary terms vanish for u with compact support. Now plugging (5.11) into (5.10), we have

$$0 = \frac{1}{2}\partial_t \int_{-\infty}^{\infty} [u_t^2 + c^2 u_x^2]\, dx.$$

Consequently, we have shown that $E'(t) = 0$.

5.4.3 Energy Methods & Domain of Dependence.

We now use the energy defined above to *prove* the finite propagation speed associated with solutions to the wave equation. Of course, d'Alembert's formula has already proven the finite speed of propagation. However, this energy method technique is useful because it can be applied to other equations.

Consider the initial value problem for the wave equation

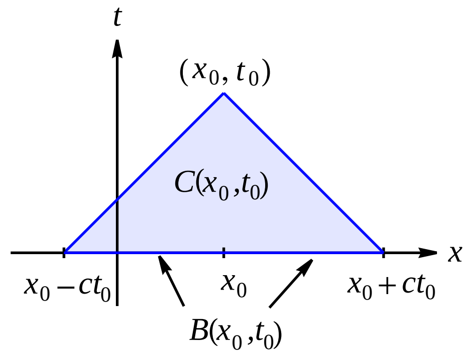
$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

Let

$$B(x_0,t_0) \equiv \{x : |x-x_0| \le ct_0\}.$$

Let

$$C(x_0,t_0) \equiv \{(x,t) : 0 \le t \le t_0, |x-x_0| \le c|t-t_0|\}$$



Theorem 3. (Finite Propagation Speed) *If* $\phi \equiv 0 \equiv \psi$ *on* $B(x_0,t_0)$*, then* $u \equiv 0$ *in* $C(x_0,t_0)$*.*

Proof. Let

$$e(t) \equiv \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2(x,t) + c^2u_x^2(x,t)\,dx \qquad 0 \le t \le t_0$$

We see that $e(t)$ is the energy of u at time t for $x \in [x_0-c(t_0-t), x_0+c(t_0-t)]$. Therefore, $e(0)$ is the energy of u at time $t=0$ for $x \in [x_0-ct_0, x_0+ct_0] = B(x_0,t_0)$. If $\phi \equiv 0 \equiv \psi \in B(x_0,t_0)$, then $e(0) \equiv 0$. Now we claim that $e(t) \le e(0)$ for all $t \in [0,t_0]$, and, therefore, $e(t) \equiv 0$. Consequently, $u \equiv 0$ in $C(x_0,t_0)$. Therefore, it remains only to show that $e(t) \le e(0)$, in other words, $e'(t) \le 0$.

For $e(t)$ as defined above, we see that

$$\begin{aligned} e(t) &= \frac{1}{2}\int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} u_t^2(x,t) + c^2u_x^2(x,t)\,dx \\ &= \frac{1}{2}\left[\int_0^{x_0+c(t_0-t)} u_t^2(x,t) + c^2u_x^2(x,t)\,dx + \int_{x_0-c(t_0-t)}^0 u_t^2(x,t) + c^2u_x^2(x,t)\,dx\right] \end{aligned}$$

which implies that

$$
\begin{aligned}
e'(t) &= \int_0^{x_0+c(t_0-t)} (u_t u_{tt} + c^2 u_x u_{xt})\,dx - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0+c(t_0-t)} \\
&\quad + \int_{x_0-c(t_0-t)}^{0} (u_t u_{tt} + c^2 u_x u_{xt})\,dx - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0-c(t_0-t)} \\
&= \int_{x_0-c(t-t_0)}^{x_0+c(t-t_0)} (u_t u_{tt} + c^2 u_x u_{xt})\,dx - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0+c(t-t_0)} \\
&\quad - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0-c(t-t_0)} \\
&= \int_{x_0-c(t-t_0)}^{x_0+c(t-t_0)} u_t(u_{tt} - c^2 u_{xx})\,dx + c^2 u_x u_t|_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0+c(t-t_0)} \\
&\quad - \frac{c}{2}(u_t^2(x,t) + c^2 u_x^2(x,t))|_{x=x_0-c(t-t_0)} \\
&= 0 + \left[-\frac{c}{2}u_t^2(x,t) + c^2 u_x u_t - \frac{c}{2}c^2 u_x^2(x,t))\right]|_{x_0+c(t-t_0)} \\
&\quad + \left[-\frac{c}{2}u_t^2(x,t) - c^2 u_x u_t - \frac{c}{2}c^2 u_x^2(x,t))\right]|_{x_0-c(t-t_0)} \\
&= -\frac{c}{2}\left[u_t^2 - 2c u_x u_t + c^2 u_x^2\right]|_{x_0+c(t-t_0)} - \frac{c}{2}\left[u_t^2 + 2c u_x u_t + c^2 u_x^2\right]_{x_0-c(t-t_0)} \\
&= -\frac{c}{2}[u_t - c u_x]^2|_{x_0+c(t-t_0)} - \frac{c}{2}[u_t + c u_x]^2|_{x_0-c(t-t_0)} \\
&\leq 0,
\end{aligned}
$$

as claimed. Therefore, $e(t) \leq e(0) = 0$. But, $e(t) \geq 0$. Therefore, $e(t) = 0$, which implies $u_t = u_x = 0$ in $C(x_0, t_0)$. Consequently, u must be a constant. But, $u = 0$ at $t = 0$. Therefore, $u = 0$ in $C(x_0, t_0)$. □

5.5 Waves with a Source

Consider the initial-value problem for the inhomogeneous wave equation on $\mathbb{R}$,

$$
\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{5.12}
$$

Theorem 4. *The unique solution of* (5.12) *is*

$$
\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.} \tag{5.13}
$$

Below we will give three different proofs of this theorem. They each provide a different way of looking at this problem.

Proof of Theorem 4: (Method 1: Reduction to First-Order Equations)

We factor our equation as

$$(\partial_t + c\partial_x)(\partial_t - c\partial_x)u = f(x,t).$$

Now let $v = (\partial_t - c\partial_x)u$. Therefore, v is a solution of the following initial-value problem for an inhomogeneous transport equation

$$\begin{cases} v_t + cv_x = f(x,t) \\ v(x,0) = u_t(x,0) - cu_x(x,0) = \psi(x) - c\phi'(x) \end{cases}$$

Now, we introduce the characteristic equations

$$\begin{aligned} \frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= c \\ \frac{dz}{ds} &= f(x,t) \end{aligned}$$

with initial conditions

$$\begin{aligned} t(r,0) &= 0 \\ x(r,0) &= r \\ z(r,0) &= \psi(r) - c\phi'(r). \end{aligned}$$

First solving our characteristic ODEs for t and x, we have

$$\begin{aligned} t(r,s) &= s \\ x(r,s) &= cs + r. \end{aligned}$$

Therefore, our equations for z become

$$\frac{dz}{ds} = f(cs + r, s) \qquad z(r,0) = \psi(r) - c\phi'(r).$$

Solving the ODE, we have

$$z(r,s) = \int_0^s f(c\widetilde{s} + r, \widetilde{s})\, d\widetilde{s} + \psi(r) - c\phi'(r).$$

Solving for r, s, we get

$$v(x,t) = \int_0^t f(c\widetilde{s} + x - ct, \widetilde{s})\, d\widetilde{s} + \psi(x - ct) - c\phi'(x - ct).$$

Now it remains to solve

$$\begin{cases} u_t - cu_x = v(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

Again, this is just an initial-value problem for an inhomogeneous transport equation. We introduce the characteristic equations,

$$\begin{aligned}\frac{dt}{ds} &= 1 \\ \frac{dx}{ds} &= -c \\ \frac{dz}{ds} &= v(x,t)\end{aligned}$$

with initial conditions

$$\begin{aligned}t(r,0) &= 0 \\ x(r,0) &= r \\ z(r,0) &= \phi(r).\end{aligned}$$

Our equations for t, x are given by

$$\begin{aligned}t(r,s) &= s \\ x(r,s) &= -cs + r.\end{aligned}$$

Therefore, our equations for z become

$$\frac{dz}{ds} = v(-cs + r, s) \qquad z(r,0) = \phi(r).$$

Solving this, we have

$$\begin{aligned}z(r,s) &= \int_0^s v(-cs' + r, s')\, ds' + \phi(r) \\ &= \int_0^s \left[\int_0^{s'} f(c\widetilde{s} + [-cs' + r] - cs', \widetilde{s})\, d\widetilde{s}\right] ds' \\ &\quad + \int_0^s \left[\psi(-cs' + r - cs') - c\phi'(-cs' + r - cs')\right] ds' + \phi(r) \\ &= \int_0^s \left[\int_0^{s'} f(c\widetilde{s} - 2cs' + r, \widetilde{s})\, d\widetilde{s}\right] ds' \\ &\quad + \int_0^s \left[\psi(-2cs' + r) - c\phi'(-2cs' + r)\right] ds' + \phi(r)\end{aligned}$$

Now solving for r, s, we have $r = x + ct$ and $s = t$. Therefore, our solution is given by

$$\begin{aligned}u(x,t) = &\int_0^t \left[\int_0^{s'} f(c\widetilde{s} - 2cs' + x + ct, \widetilde{s})\, d\widetilde{s}\right] ds' \\ &+ \int_0^t \left[\psi(-2cs' + x + ct) - c\phi'(-2cs' + x + ct)\right] ds' + \phi(x + ct).\end{aligned} \tag{5.14}$$

Now we need to look at each of the four terms on the right-hand side above. First, making the change of variable $y = -2cs' + x + ct$, we have

$$\begin{aligned}
\int_0^t \psi(-2cs' + x + ct)\, ds' &= \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy \\
-c\int_0^t \phi'(-2cs' + x + ct)\, ds' &= -\frac{1}{2}\int_{x-ct}^{x+ct} \phi'(y)\, dy \\
&= \frac{1}{2}\phi(x-ct) - \frac{1}{2}\phi(x+ct).
\end{aligned}$$

Therefore, the last three terms in (5.14) are just

$$\frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\, dy.$$

Therefore, it remains to look at the first term in (5.14). Making the change of variables $y = c\widetilde{s} - 2cs' + x + ct$ and $s = \widetilde{s}$, we have

$$\int_0^t \int_0^{s'} f(c\widetilde{s} - 2cs' + x + ct, \widetilde{s})\, d\widetilde{s}\, ds' = \int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s) J\, dy\, ds,$$

where J is the Jacobian of the change of variables. That is,

$$\begin{aligned}
J &= \left| \det \begin{bmatrix} \widetilde{s}_y & \widetilde{s}_s \\ s'_y & s'_s \end{bmatrix} \right| \\
&= \left| \det \begin{bmatrix} 0 & 1 \\ -\frac{1}{2c} & \frac{1}{2} \end{bmatrix} \right| \\
&= \frac{1}{2c}.
\end{aligned}$$

Therefore, the first term in (5.14) is

$$\frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\, dy\, ds.$$

Therefore, our theorem is proved.

□

Proof of Theorem 4: (Method 2: Using Green's Theorem)

Ref: Strauss, Section 2.4

Fix a point (x_0, t_0). Let $\Delta = \{(x,t) : 0 \le t \le t_0; |x - x_0| \le c|t - t_0|\}$, the domain of dependence for the point (x_0, t_0). Integrating the wave equation

$$u_{tt} - c^2 u_{xx} = f(x,t)$$

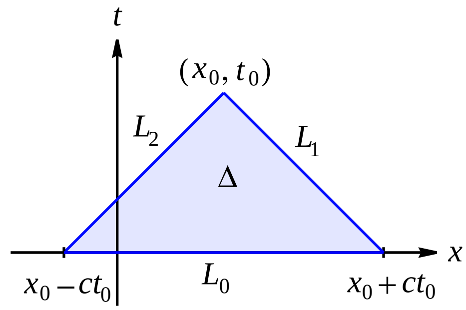
over Δ, we have

$$\iint_\Delta u_{tt} - c^2 u_{xx}\, dx\, dt = \iint_\Delta f(x,t)\, dx\, dt. \tag{5.15}$$

By Green's theorem, we know

$$\iint_\Delta \left(\frac{\partial P}{\partial x} - \frac{\partial Q}{\partial t}\right) dx\,dt = \int_{\partial\Delta} P\,dt + Q\,dx$$

where $\partial\Delta$ is the boundary of Δ traversed in the counterclockwise direction.



Therefore, we have

$$\begin{aligned} -\iint_\Delta \left((c^2u_x)_x - (u_t)_t\right)\,dx\,dt &= -\int_{\partial\Delta} \left(c^2u_x\,dt + u_t\,dx\right) \\ &= -\sum_{i=0}^{2}\int_{L_i}\left(c^2u_x\,dt + u_t\,dx\right). \end{aligned}$$

Now, first

$$\begin{aligned} -\int_{L_0}(c^2u_x\,dt + u_t\,dx) &= -\int_{x_0-ct_0}^{x_0+ct_0} u_t(x,0)\,dx \\ &= -\int_{x_0-ct_0}^{x_0+ct_0}\psi(x)\,dx. \end{aligned} \tag{5.16}$$

Next, on L_1, $\frac{dx}{dt} = -c$. Therefore,

$$\begin{aligned} -\int_{L_1}(c^2u_x\,dt + u_t\,dx) &= c\int_{L_1}(u_x\,dx + u_t\,dt) \\ &= c\int_{L_1} du \\ &= c[u(x_0,t_0) - u(x_0+ct_0,0)] \\ &= c[u(x_0,t_0) - \phi(x_0+ct_0)], \end{aligned} \tag{5.17}$$

while on L_2, $\frac{dx}{dt} = c$. Therefore,

$$\begin{aligned} -\int_{L_2}(c^2u_x\,dt + u_t\,dx) &= -c\int_{L_2}(u_x\,dx + u_t\,dt) \\ &= -c\int_{L_2} du \\ &= -c[u(x_0-ct_0,0) - u(x_0,t_0)] \\ &= -c[\phi(x_0-ct_0) - u(x_0,t_0)]. \end{aligned} \tag{5.18}$$

Therefore, combining (5.16), (5.17) and (5.18) with (5.15), we conclude that

$$2cu(x_0, t_0) - c\phi(x_0 + ct_0) - c\phi(x_0 - ct_0) - \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx = \iint_\Delta f(x,t)\,dx\,dt.$$

Consequently, we conclude

$$u(x_0, t_0) = \frac{1}{2}[\phi(x_0 + ct_0) + \phi(x_0 - ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)\,dx + \frac{1}{2c}\int_0^t \int_{x_0-c(t_0-t)}^{x_0+c(t_0-t)} f(x,t)\,dx\,dt,$$

as desired. □

Proof of Theorem 4: (Method 3: Operator Method (Duhamel's Principle))

Motivation

We start by reviewing some ideas from ODE theory.

First-Order ODE

Consider the first-order inhomogeneous ODE

$$\begin{cases} u'(t) + au(t) = f(t) \\ u(0) = \phi. \end{cases} \tag{5.19}$$

Multiplying both sides of the ODE by the integrating factor e^{at}, we have

$$(e^{at}u)' = e^{at}f(t)$$

and therefore,

$$e^{at}u = \int_0^t e^{as}f(s)\,ds + C,$$

which implies

$$u(t) = e^{-at}\int_0^t e^{as}f(s)\,ds + Ce^{-at}.$$

Then, substituting in $t = 0$, we see that

$$u(t) = \phi e^{-at} + \int_0^t e^{-a(t-s)}f(s)\,ds$$

is the solution of the initial-value problem (5.19). Defining the *solution operator*

$$S(t)\phi = e^{-at}\phi,$$

we see that

$$u(t) = S(t)\phi + \int_0^t S(t-s)f(s)\,ds.$$

Notice that $u_h(t) \equiv S(t)\phi$ is the solution of the homogeneous initial-value problem

$$\begin{cases} u'(t) + au(t) = 0 \\ u(0) = \phi \end{cases}$$

This technique of using the solution operator associated with the linear equation to solve the inhomogeneous problem is known as **Duhamel's principle**.

We now try the same technique with a second-order ODE.

Second-Order ODE

Consider the second-order inhomogeneous ODE

$$\begin{cases} u''(t) + a^2 u(t) = f(t) \\ u(0) = \phi \\ u_t(0) = \psi. \end{cases} \tag{5.20}$$

We can write this equation as a system, by introducing a new function v, such that

$$u_t = av$$
$$v_t = -au + \frac{1}{a} f$$

or, in matrix form as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{a} f \end{bmatrix}$$

which can be rewritten as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t + \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{a} f \end{bmatrix}.$$

Letting

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad F = \begin{bmatrix} 0 \\ \frac{1}{a} f \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix},$$

our equation can be written as

$$U_t + AU = F.$$

Our initial conditions $u(0) = \phi$, $u'(0) = \psi$ imply

$$U(0) = \begin{bmatrix} u(0) \\ v(0) \end{bmatrix} = \begin{bmatrix} u(0) \\ \frac{1}{a} u'(0) \end{bmatrix} = \begin{bmatrix} \phi \\ \frac{1}{a} \psi \end{bmatrix}.$$

Defining

$$\Phi = \begin{bmatrix} \phi \\ \frac{1}{a} \psi \end{bmatrix},$$

we can rewrite our initial-value problem (5.20) as

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi. \end{cases} \tag{5.21}$$

Multiplying the ODE in (5.21) by the matrix exponential $e^{At} = \sum_n \frac{(At)^n}{n!}$, our equation becomes

$$(e^{At} U)_t = e^{At} F,$$

which implies

$$U = e^{-At} \int_0^t e^{As} F(s)\, ds + e^{-At} C.$$

Our initial condition $U(0) = \Phi$ implies our unique solution of (5.21) is given by

$$U(t) = e^{-At}\Phi + \int_0^t e^{-A(t-s)} F(s)\, ds.$$

Defining the solution operator

$$S(t)W = e^{-At}W,$$

we see that our solution is given by

$$U(t) = S(t)\Phi + \int_0^t S(t-s)F(s)\, ds.$$

Again, notice that $S(t)\Phi = e^{-At}\Phi$ is the solution of the homogeneous initial-value problem,

$$\begin{cases} U_t + AU = 0 \\ U(0) = \Phi. \end{cases}$$

Consequently, we have been able to write the solution of the inhomogeneous problem using the solution operator for the homogeneous problem. We now try to extend these ideas to solving inhomogeneous partial differential equations. In particular, we look here at the wave equation.

We return to the initial value problem for the wave equation,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u(x,0) = \phi \\ u_t(x,0) = \psi. \end{cases} \tag{5.22}$$

Using the ideas from the case of the second-order ODE described above, we begin by writing the wave equation as a system. In particular, letting $v = u_t$, we can write the inhomogeneous wave equation as

$$\begin{aligned} u_t &= v \\ v_t &= c^2 u_{xx} + f, \end{aligned}$$

which can be written in matrix form as

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ c^2\partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

Letting

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2\partial_x^2 & 0 \end{bmatrix} \qquad F = \begin{bmatrix} 0 \\ f \end{bmatrix},$$

our equation can be written in matrix form as

$$U_t + AU = F,$$

where A is an operator matrix. Our initial conditions $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$ imply

$$U(x,0) = \begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} u(x,0) \\ u_t(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}.$$

Defining

$$\Phi \equiv \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix},$$

our initial-value problem (5.22) can be rewritten in matrix form as

$$\begin{cases} U_t + AU = F \\ U(x,0) = \Phi. \end{cases} \tag{5.23}$$

Our hope is that if we can find the solution operator $S(t)$ for the homogeneous problem, then the solution of the inhomogeneous problem (5.23) will be given by

$$U(x,t) = S(t)\Phi + \int_0^t S(t-s)F(s)\,ds.$$

Therefore, we start by considering the homogeneous problem

$$\begin{cases} U_t + AU = 0 \\ U(x,0) = \Phi. \end{cases} \tag{5.24}$$

Of course, this is just the initial-value problem for the homogeneous wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

By d'Alembert's formula (5.8), we know the solution is given by

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy.$$

Therefore, the solution of (5.24) is given by

$$U(x,t) = \begin{bmatrix} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \\ \frac{c}{2}[\phi'(x+ct) - \phi'(x-ct)] + \frac{1}{2}[\psi(x+ct) + \psi(x-ct)] \end{bmatrix}.$$

In other words, defining the solution operator $S(t)$ as

$$\begin{aligned} S(t)\Phi &= S(t)\begin{bmatrix} \phi \\ \psi \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \\ \frac{c}{2}[\phi'(x+ct) - \phi'(x-ct)] + \frac{1}{2}[\psi(x+ct) + \psi(x-ct)] \end{bmatrix}, \end{aligned} \tag{5.25}$$

we see that the solution of (5.24) is given by $S(t)\Phi$.

As stated above, our conjecture is that the solution of the inhomogeneous initial-value problem (5.23) is given by

$$U(x,t) = S(t)\Phi + \int_0^t S(t-s)F(s)\,ds, \tag{5.26}$$

with $S(t)$ as defined in (5.25). Looking at the first component of this vector-valued equation (5.26), we see this would imply

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy + \int_0^t \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds.$$

Of course, as we have already shown by two other methods, this is indeed the unique solution of the initial-value problem for the inhomogeneous wave equation (5.12).

□

Duhamel's Principle

The nice property of using this *solution operator* technique, also known as *Duhamel's principle* is that it extends to much more general PDEs. Consider a general initial-value problem for an *evolution equation* of the form

$$\begin{cases} U_t + AU = F \\ U(\vec{x},0) = \Phi(\vec{x}), \end{cases} \tag{5.27}$$

where $\vec{x} \in \mathbb{R}^m$, $U : \mathbb{R}^m \times [0,\infty) \to \mathbb{R}^n$, $F : \mathbb{R}^m \times [0,\infty) \to \mathbb{R}^n$ and A is an $n \times n$ linear operator matrix independent of t.

Suppose $U_h(\vec{x},t) = S(t)\Phi(\vec{x})$ is the solution of the homogeneous initial-value problem

$$\begin{cases} U_t + AU = 0 \\ U(\vec{x},0) = \Phi(\vec{x}). \end{cases} \tag{5.28}$$

Then the solution of the inhomogeneous problem (5.27) is given by

$$U(\vec{x},t) = S(t)\Phi(\vec{x}) + \int_0^t S(t-s)F(\vec{x},s)\,ds. \tag{5.29}$$

To be precise about this statement, we would need to deal with convergence issues of the integrals, etc. So, here, we just provide a formal proof of this statement. First, we show that u defined in (5.29) satisfies our initial condition. By assumption,

$$U_h(\vec{x},t) = S(t)\Phi(\vec{x})$$

satisfies (5.28). Therefore, $U_h(\vec{x},0) = S(0)\Phi(\vec{x}) = \Phi(\vec{x})$. Consequently, for U defined in (5.29), $U(x,0) = S(0)\Phi(\vec{x}) + 0 = \Phi(\vec{x})$. Next, we need to show that U defined in (5.29) satisfies the inhomogeneous PDE. We have

$$\begin{aligned}
U_t &= (S(t)\Phi(\vec{x}))_t + S(0)F(\vec{x},t) + \int_0^t (S(t-s)F(\vec{x},s))_t\,ds \\
&= -AS(t)\Phi(\vec{x}) + F(\vec{x},t) - \int_0^t AS(t-s)F(\vec{x},s)\,ds \\
&= F(\vec{x},t) - A\left[S(t)\Phi(\vec{x}) + \int_0^t S(t-s)F(\vec{x},s)\,ds\right] \\
&= F(\vec{x},t) - AU.
\end{aligned}$$

Therefore, we conclude that

$$U_t + AU = F(\vec{x},t),$$

as desired.

5.6 Reflections of Waves

Ref: Strauss, Section 3.2

5.6.1 The Wave Equation on a Half-Line

Consider the following *Dirichlet problem* on the half-line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < +\infty \\ u(x,0) = \phi(x) & x > 0 \\ u_t(x,0) = \psi(x) & x > 0 \\ u(0,t) = 0 & t \geq 0. \end{cases} \tag{5.30}$$

We look for a solution by extending the functions $\phi(x)$ and $\psi(x)$ to all of $\mathbb{R}$ by *odd reflection*. That is, let

$$\phi_{\text{odd}}(x) = \begin{cases} \phi(x) & x \geq 0 \\ -\phi(-x) & x \leq 0 \end{cases}$$

And, similarly, let

$$\psi_{\text{odd}}(x) = \begin{cases} \psi(x) & x \geq 0 \\ -\psi(-x) & x \leq 0 \end{cases}$$

Let $\tilde{u}$ be the solution of the initial-value problem on the whole real line with initial data $\phi_{\text{odd}}(x)$ and $\psi_{\text{odd}}(x)$,

$$\begin{cases} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} = 0 & -\infty < x < \infty \\ \tilde{u}(x,0) = \phi_{\text{odd}}(x) \\ \tilde{u}_t(x,0) = \psi_{\text{odd}}(x). \end{cases}$$

Let $u(x,t) \equiv \tilde{u}(x,t)$ for $0 < x < \infty$ (restriction of $\tilde{u}$ to the half-line). *Claim:* $u(x,t)$ is the solution to the Dirichlet problem on the half-line (5.30).

Clearly, u satisfies the PDE for $0 < x < \infty$, as $u = \tilde{u}$ and $\tilde{u}$ satisfies the PDE on the whole real line. It just remains to check that u satisfies the initial conditions. But, for $x > 0$,

$u(x,0) = \tilde{u}(x,0) = \phi_{\text{odd}}(x) = \phi(x)$ and $u_t(x,0) = \tilde{u}_t(x,0) = \psi_{\text{odd}}(x) = \psi(x)$. Therefore, $u(x,t) \equiv \tilde{u}(x,t)$ for $x > 0$ is the solution of (5.30).

Now, we look for a formula similar to d'Alembert's formula for the solution of this problem. From d'Alembert's formula, we know $\tilde{u}$ is given by

$$\tilde{u}(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(y)\,dy.$$

For $t > 0$, if $x > ct$, then $\phi_{\text{odd}}(x-ct) = \phi(x-ct)$ and $\phi_{\text{odd}}(x+ct) = \phi(x+ct)$. Therefore, for $t > 0$, $x > ct$, our solution is given by

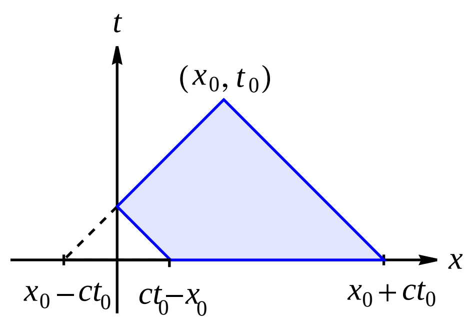
$$\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy, \qquad \text{for } t > 0, x > ct.} \tag{5.31}$$

Now consider $t > 0$, $x < ct$, then $\phi_{\text{odd}}(x-ct) = -\phi(ct-x)$, and $\psi_{\text{odd}}(y) = -\psi(-y)$ for $y < 0$. Therefore,

$$\begin{aligned}
\frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\,dy &= \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_{x-ct}^{0} \psi(y)\,dy \\
&= \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy - \frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,dy \\
&= \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy.
\end{aligned}$$

Therefore, for $t > 0$, $x < ct$, our formula becomes,

$$\boxed{u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(y)\,dy, \qquad \text{for } t > 0, x < ct.} \tag{5.32}$$



Inhomogeneous Wave Equation on a Half-Line

We now consider the inhomogeneous wave equation on a half-line with Dirichlet boundary conditions,

$$\begin{cases}
u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < \infty \\
u(x,0) = \phi(x) & 0 < x < \infty \\
u_t(x,0) = \psi(x) & 0 < x < \infty \\
u(0,t) = 0 & t \geq 0.
\end{cases} \tag{5.33}$$

As described above, we extend the initial data to be odd. Here we also extend $f(x,t)$ to be odd about the line $x=0$ for all t. Now consider the wave equation on the whole line

$$\begin{cases} u_{tt} - c^2 u_{xx} = f_{\text{odd}}(x,t) \\ u(x,0) = \phi_{\text{odd}}(x) \\ u_t(x,0) = \psi_{\text{odd}}(x). \end{cases} \tag{5.34}$$

Now the unique solution $\widetilde{u}$ of (5.34) is given by d'Alembert's formula (5.8) as

$$\begin{aligned} \widetilde{u}(x,t) = \frac{1}{2}[\phi_{\text{odd}}(x+ct) + \phi_{\text{odd}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{odd}}(y)\,dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{\text{odd}}(y,s)\,dy\,ds. \end{aligned} \tag{5.35}$$

Now we claim that $u(x,t) = \widetilde{u}(x,t)$ for $x \geq 0$ is the unique solution of (5.33). Clearly u satisfies the PDE for $x > 0$. In addition, u satisfies the initial conditions (as described in the previous case). Therefore, the only thing that it remains to check is that u satisfies the boundary condition $u(0,t) = 0$ for all $t > 0$. This would certainly be true if $\widetilde{u}(x,t)$ was an odd function in x. But, this can be seen easily from (5.35). Therefore, u solves (5.33). It is left as an exercise to verify that u is the *unique* solution.

Now for a point (x_0, t_0) with $t_0 > 0$ and $x_0 > ct_0$, then our solution formula is

$$\boxed{\begin{aligned} u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) + \phi(x_0-ct_0)] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(y)\,dy \\ + \frac{1}{2c}\int_0^{t_0} \int_{x_0-c(t_0-s)}^{x_0+c(t_0-s)} f(y,s)\,dy\,ds \qquad t_0 > 0, x_0 > ct_0. \end{aligned}} \tag{5.36}$$

Therefore, the domain of dependence is the cone $\{(x,t) : t \geq 0; |x - x_0| \leq c|t - t_0|\}$. Alternatively, if (x_0, t_0) is a point such that $t_0 > 0$, $x_0 < ct_0$, then our solution is given by

$$\boxed{\begin{aligned} u(x_0,t_0) = \frac{1}{2}[\phi(x_0+ct_0) - \phi(ct_0-x_0)] + \frac{1}{2c}\int_{ct_0-x_0}^{x_0+ct_0} \psi(y)\,dy \\ + \frac{1}{2c}\int_0^{t_0} \int_{c(t_0-s)-x_0}^{x_0+c(t_0-s)} f(y,s)\,dy\,ds \qquad t_0 > 0, x_0 < ct_0. \end{aligned}} \tag{5.37}$$

Therefore, the domain of dependence is the region shown in the figure below (5.32).

Remark: Alternatively, we could derive the solution formula for (5.33) using Duhamel's principle. As before, defining

$$\begin{aligned} U &= \begin{bmatrix} u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2\partial_x^2 & 0 \end{bmatrix} \\ F &= \begin{bmatrix} 0 \\ f \end{bmatrix} \qquad \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}, \end{aligned}$$

we can write (5.33) as the system

$$\begin{cases} U_t + AU = F & 0 < x < \infty \\ U(x,0) = \Phi(x) & 0 < x < \infty \\ U(0,t) = \begin{bmatrix} 0 \\ v(0,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & . \end{cases} \tag{5.38}$$

From (5.31) and (5.32), we know the solution of the homogeneous system

$$\begin{cases} U_t + AU = 0 & 0 < x < \infty \\ U(x,0) = \Phi(x) & 0 < x < \infty \\ U(0,t) = \begin{bmatrix} 0 \\ v(0,t) \end{bmatrix} \end{cases} \tag{5.39}$$

in terms of ϕ and ψ. (at least for $t > 0$; the case $t < 0$ can be handled similarly) As a result, we can write the solution of (5.39) in terms of a solution operator $S(t)$ defined as follows. For $t > 0$, $x > ct$, we define $S(t)$ such that

$$S(t)\Phi(x) = \begin{bmatrix} \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \\ \frac{c}{2}[\phi'(x+ct) - \phi'(x-ct)] + \frac{1}{2}[\psi(x+ct) + \psi(x-ct)] \end{bmatrix} \qquad t > 0, x > ct.$$

For $t > 0$, $x < ct$, we define $S(t)$ such that

$$S(t)\Phi(x) = \begin{bmatrix} \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)\,dy \\ \frac{c}{2}[\phi'(x+ct) - \phi'(ct-x)] + \frac{1}{2}[\psi(x+ct) - \psi(ct-x)] \end{bmatrix} \qquad t > 0, x < ct.$$

By Duhamel's principle, the solution of (5.38) "should" be

$$U(x,t) = S(t)\Phi(x) + \int_0^t S(t-s)F(x,s)\,ds.$$

While we have verified that a function defined in this way will satisfy the inhomogenous PDE and the initial conditions, we should also verify that this function will satisfy our boundary conditions. In fact, this is true. We leave it to the reader to verify. Consequently, the solution of (5.33) will be given by the first component of $U(x,t)$. Using the definitions of $S(t)$ above, we see that the solution of (5.33) is given by (5.36) and (5.37) in the regions $t > 0, x > ct$ and $t > 0, x < ct$ respectively.

5.6.2 The Finite Interval

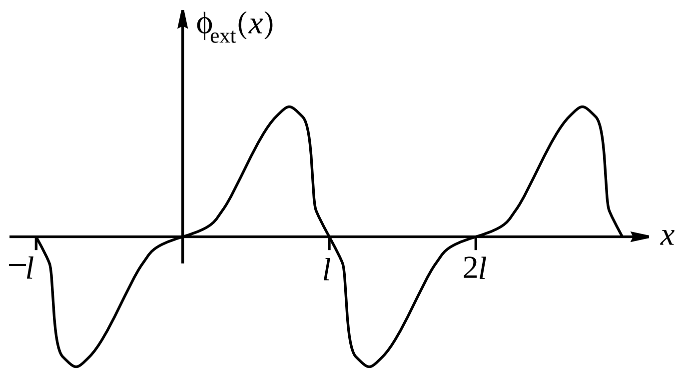
Now consider the wave equation on a finite interval which satisfies Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = \phi(x) & 0 < x < l \\ u_t(x,0) = \psi(x) & 0 < x < l \\ u(0,t) = 0 = u(l,t). \end{cases} \tag{5.40}$$

Motivated by the previous section, we extend our initial data $\phi(x)$ and $\psi(x)$ to be odd with respect to both $x = 0$ and $x = l$. In particular, define

$$\phi_{\text{ext}}(x) = \begin{cases} \phi(x) & 2nl < x < (2n+1)l \\ -\phi(-x) & (2n+1)l < x < 2nl \end{cases}$$

Similarly, define $\psi_{\text{ext}}(x)$.



Let $\tilde{u}(x,t)$ be the solution of

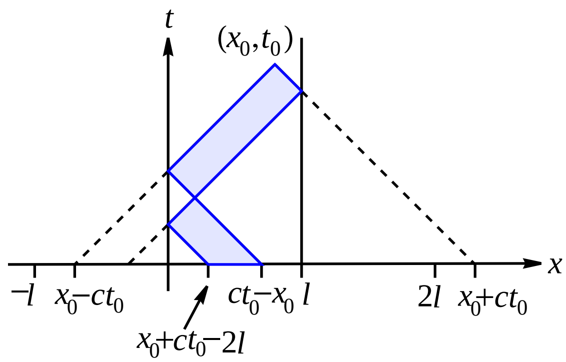
$$\begin{cases} \tilde{u}_{tt} - c^2\tilde{u}_{xx} = 0, & -\infty < x < \infty \\ \tilde{u}(x,0) = \phi_{\text{ext}}(x) \\ \tilde{u}_t(x,0) = \psi_{\text{ext}}(x). \end{cases}$$

We know $\widetilde{u}$ is given by d'Alembert's formula (5.8) as

$$\widetilde{u}(x,t) = \frac{1}{2}[\phi_{\text{ext}}(x+ct) + \phi_{\text{ext}}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{\text{ext}}(y)\,dy \tag{5.41}$$

Let $u(x,t) \equiv \tilde{u}(x,t)$ for $0 \leq x \leq l$. We claim u is the unique solution of (5.40). To prove this claim, first we note that u clearly satisfies the PDE on the interval $0 < x < l$, as $\tilde{u}$ satisfies the PDE on the whole real line. Next, we check that the initial conditions are satisfied. But, for $0 < x < l$, $u(x,0) = \tilde{u}(x,0) = \phi_{\text{ext}}(x) = \phi(x)$ and $u_t(x,0) = \tilde{u}_t(x,0) = \psi_{\text{ext}}(x) = \psi(x)$. Therefore, the initial conditions are satisfied. Last, we show that the boundary conditions are satisfied. From the formula for $\widetilde{u}(x,t)$ (5.41), we see that $\widetilde{u}(x,t)$ is odd with respect to $x = 0$ and $x = l$ for all $t \geq 0$. Therefore, $\widetilde{u}(0,t) = 0 = \widetilde{u}(l,t)$ for all $t \geq 0$ which implies $u(0,t) = 0 = u(l,t)$ for all $t \geq 0$, as desired.

Now we can use (5.41) to find the solution $u(x,t)$ of (5.40) at any point (x,t) in terms of the initial data ϕ, ψ. Consider the following example. Let (x_0, t_0) be the point shown below. Then using (5.41), we have a formula for $u(x_0,t_0)$ in terms of ϕ_{ext} and ψ_{ext}. We can write this in terms of ϕ and ψ as follows.



First, we see that

$$-l < x_0 - ct_0 < 0$$
$$2l < x_0 + ct_0 < 3l.$$

Using the fact that ϕ_{ext} is $2l$-periodic, we know

$$\phi_{\text{ext}}(x_0 - ct_0) = -\phi_{\text{ext}}(ct_0 - x_0) = -\phi(ct_0 - x_0),$$

while

$$\phi_{\text{ext}}(x_0 + ct_0) = \phi_{\text{ext}}(x_0 + ct_0 - 2l) = \phi(x_0 + ct_0 - 2l).$$

In addition,

$$\begin{aligned}
\int_{x_0-ct_0}^{x_0+ct_0} \psi_{\text{ext}}(y)\,dy &= \int_{x_0-ct_0}^{0} \psi_{\text{ext}}(y)\,dy + \int_0^l \psi_{\text{ext}}(y)\,dy + \int_l^{2l} \psi_{\text{ext}}(y)\,dy + \int_{2l}^{x_0+ct_0} \psi_{\text{ext}}(y)\,dy \\
&= -\int_0^{ct_0-x_0} \psi(y)\,dy + \int_0^l \psi(y)\,dy - \int_0^l \psi(y)\,dy + \int_0^{x_0+ct_0-2l} \psi(y)\,dy \\
&= -\int_{x_0+ct_0-2l}^{ct_0-x_0} \psi(y)\,dy.
\end{aligned}$$

Therefore, for (x_0, t_0) in the region shown, our solution is given by

$$u(x_0, t_0) = \frac{1}{2}[\phi(x_0 + ct_0 - 2l) - \phi(ct_0 - x_0)] - \frac{1}{2c}\int_{x_0+ct_0-2l}^{ct_0-x_0} \psi(y)\,dy.$$

Moreover, the domain of dependence of the solution is given by the region shown.

Remarks.

- This technique can also be used to study the inhomogeneous wave equation on an interval with Dirichlet boundary conditions by extending the source function to be odd with respect to $x = 0$ and $x = l$, or using Duhamel's principle. We remark that the domain of dependence for the inhomogeneous problem will be the same as the domain of dependence for the corresponding homogeneous problem.

- While this reflection technique allows one to get an explicit formula for the solution u of the wave equation on a finite interval at a specific point (x_0, t_0), it is somewhat tedious. In the next section, we introduce the separation of variables technique which allows you to represent solutions in terms of infinite series. While having the drawback of providing you only with a series solution, it is easier and more versatile than the reflection technique described here and consequently, used much more often.