7 Wave Equation in Higher Dimensions

We now consider the initial-value problem for the wave equation in n dimensions,

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases} \tag{7.1}$$

where $\Delta u \equiv \sum_{i=1}^{n} u_{x_i x_i}$.

7.1 Method of Spherical Means

Ref: Evans, Sec. 2.4.1; Strauss, Sec. 9.2

We begin by introducing a method to solve (7.1) in odd dimensions. First, we introduce some notation. For $x \in \mathbb{R}^n$, let

- $B(x,r)$ = Ball of radius r about x
- $\partial B(x,r)$ = Boundary of ball of radius r about x
- $\alpha(n)$ = Volume of unit ball in $\mathbb{R}^n$
- $n\alpha(n)$ = Surface Area of unit ball in $\mathbb{R}^n$.

With this notation, the volume of the ball of radius r about $x \in \mathbb{R}^n$, written as $\text{Vol}(B(x,r))$, is given by $\alpha(n)r^n$ and the surface area of the ball of radius r about $x \in \mathbb{R}^n$, written as $\text{S.A.}(B(x,r))$, is given by $n\alpha(n)r^{n-1}$.

For $f : \mathbb{R}^n \to \mathbb{R}$, we define the **average of f over $\mathbf{B(x,r)}$** as

$$⨍_{B(x,r)} f(y)\,dy \equiv \frac{1}{\text{Vol}(B(x,r))} \int_{B(x,r)} f(y)\,dy = \frac{1}{\alpha(n)r^n} \int_{B(x,r)} f(y)\,dy.$$

We define the **average of f over $\partial\mathbf{B(x,r)}$** as

$$⨍_{\partial B(x,r)} f(y)\,dS(y) \equiv \frac{1}{\text{S.A.}(B(x,r))} \int_{\partial B(x,r)} f(y)\,dS(y) = \frac{1}{n\alpha(n)r^{n-1}} \int_{\partial B(x,r)} f(y)\,dS(y),$$

where $dS(y)$ denotes the surface measure of $B(x,r)$ in $\mathbb{R}^n$.

Example 1. For $n = 3$, $\text{Vol}(B(x,r)) = \frac{4}{3}\pi r^3$. Therefore, for $f : \mathbb{R}^3 \to \mathbb{R}$, the average of f over $B(0,r)$ is given by

$$⨍_{B(0,r)} f(y)\,dy = \frac{3}{4\pi r^3} \int_0^{\pi} \int_0^{2\pi} \int_0^{r} f(\rho,\theta,\phi)\rho^2 \sin\phi\,d\rho\,d\theta\,d\phi.$$

For $n = 3$, $\text{S.A.}(B(x,r)) = 4\pi r^2$. Therefore, for $f : \mathbb{R}^3 \to \mathbb{R}$, the average of f over $\partial B(0,r)$ is given by

$$⨍_{\partial B(0,r)} f(y)\,dS(y) = \frac{1}{4\pi r^2} \int_0^{\pi} \int_0^{2\pi} f(r,\theta,\phi) r^2 \sin\phi\,d\theta\,d\phi = \frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} f(r,\theta,\phi) \sin\phi\,d\theta\,d\phi.$$

◇

Our plan to solve (7.1) is the following. Fix a point $x \in \mathbb{R}^n$. For $r > 0$, we define

$$\overline{u}(x;r,t) \equiv \fint_{\partial B(x,r)} u(y,t)\,dS(y),$$

the average of $u(\cdot,t)$ over $\partial B(x,r)$. For $r = 0$, we define $\overline{u}(x;0,t) = u(x,t)$. For $r < 0$, we define $\overline{u}(x;r,t) = \overline{u}(x;-r,t)$. We claim that for u smooth, $\overline{u}$ is a continuous function of r, and, therefore,

$$\lim_{r\to 0+} \overline{u}(x;r,t) = u(x,t).$$

In order to solve (7.1), we will assume u is a solution of (7.1) and look for an equation $\overline{u}$ solves. *Note:* We will assume $c = 1$. For $c \neq 1$, we can make a change of variables to derive the solution from the solution in the case $c = 1$.

Lemma 2. *If u solves*

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^n, t \geq 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x), \end{cases}$$

then $\overline{u}(x;r,t)$ solves

$$\begin{cases} \overline{u}_{tt} - \overline{u}_{rr} - \dfrac{(n-1)}{r}\overline{u}_r = 0, & 0 < r < \infty, t \geq 0 \\ \overline{u}(x;r,0) = \overline{\phi}(x;r) \equiv \fint_{\partial B(x,r)} \phi(y)\,dS(y) \\ \overline{u}_t(x;r,0) = \overline{\psi}(x;r) \equiv \fint_{\partial B(x,r)} \psi(y)\,dS(y) \end{cases}$$

for every $x \in \mathbb{R}^n$.

Proof.

$$\begin{aligned} \overline{u}(x;r,t) &= \fint_{\partial B(x,r)} u(y,t)\,dS(y) \\ &= \fint_{\partial B(0,1)} u(x+rz,t)\,dS(z). \end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{u}_r(x;r,t) &= \fint_{\partial B(0,1)} \nabla u(x+rz,t)\cdot z\,dS(z)\\
&= \fint_{\partial B(x,r)} \nabla u(y,t)\cdot \frac{y-x}{r}\,dS(y)\\
&= \fint_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y,t)\,dS(y)\\
&= \frac{1}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} \frac{\partial u}{\partial \nu}(y,t)\,dS(y)\\
&= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} \Delta u(y,t)\,dy\\
&= \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} u_{tt}(y,t)\,dy
\end{aligned}$$

by the Divergence Theorem, and using the fact that u solves the wave equation, $u_{tt}-\Delta u = 0$. Therefore,

$$\overline{u}_r(x;r,t) = \frac{1}{n\alpha(n)r^{n-1}}\int_{B(x,r)} u_{tt}(y,t)\,dy$$

which implies

$$r^{n-1}\overline{u}_r(x;r,t) = \frac{1}{n\alpha(n)}\int_{B(x,r)} u_{tt}(y,t)\,dy.$$

Therefore,

$$\begin{aligned}
(r^{n-1}\overline{u}_r(x;r,t))_r &= \frac{1}{n\alpha(n)}\int_{\partial B(x,r)} u_{tt}(y,t)\,dS(y)\\
&= \frac{r^{n-1}}{n\alpha(n)r^{n-1}}\int_{\partial B(x,r)} u_{tt}(y,t)\,dS\\
&= r^{n-1}\fint_{\partial B(x,r)} u_{tt}(y,t)\,dS(y)\\
&= r^{n-1}\overline{u}_{tt}(x;r,t).
\end{aligned}$$

Therefore,

$$(r^{n-1}\overline{u}_r(x;r,t))_r = r^{n-1}\overline{u}_{tt}(x;r,t),$$

which implies

$$(n-1)r^{n-2}\overline{u}_r + r^{n-1}\overline{u}_{rr} = r^{n-1}\overline{u}_{tt}.$$

Therefore,

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{(n-1)}{r}\overline{u}_r = 0$$

and

$$\overline{u}(x;r,0) = \fint_{\partial B(x,r)} u(y,0)\,dS = \fint_{\partial B(x,r)} \phi(y)\,dS = \overline{\phi}(x;r).$$

Similarly,

$$\overline{u}_t(x;r,0) = \overline{\psi}(x;r)$$

as claimed. □

Solution for n = 3.

We now consider the case of the wave equation in three dimensions. Assume u is a solution of (7.1) for $n = 3$. As before define the function $\overline{u}(x;r,t)$ such that

$$\overline{u}(x;r,t) = \fint_{\partial B(x,r)} u(y,t)\,dS(y).$$

Next introduce a function $v(x;r,t)$ such that

$$v(x;r,t) = r\overline{u}(x;r,t)$$

and new functions $g(x;r)$ and $h(x;r)$ such that

$$\begin{aligned} g(x;r) &= r\overline{\phi}(x;r) = r \fint_{\partial B(x,r)} \phi(r)\,dS(y) \\ h(x;r) &= r\overline{\psi}(x;r) = r \fint_{\partial B(x,r)} \psi(r)\,dS(y). \end{aligned}$$

Lemma 3. *For each $x \in \mathbb{R}^n$, the function $v(x;r,t)$ solves the one-dimensional wave equation on the half-line with Dirichlet boundary conditions,*

$$\begin{cases} v_{tt} - v_{rr} = 0 & 0 < r < \infty, t \geq 0 \\ v(x;r,0) = g(x;r) & 0 < r < \infty \\ v_t(x;r,0) = h(x;r) & 0 < r < \infty \\ v(x;0,t) = 0 & t \geq 0. \end{cases}$$

Proof.

$$\begin{aligned} v_{tt} &= r\overline{u}_{tt} \\ &= r\left[\overline{u}_{rr} + \frac{2}{r}\overline{u}_r\right] \\ &= r\overline{u}_{rr} + 2\overline{u}_r \\ &= (r\overline{u}_r + \overline{u})_r \\ &= (r\overline{u})_{rr} \\ &= v_{rr}. \end{aligned}$$

Next,

$$\begin{aligned}
v(x;r,0) &= r\overline{u}(x;r,0) \\
&= r \fint_{\partial B(x,r)} u(y,0)\, dS(y) \\
&= r \fint_{\partial B(x,r)} \phi(y)\, dS(y) \\
&= r\overline{\phi}(x,r) \\
&= g(x;r)
\end{aligned}$$

Similarly,

$$v_t(x;r,0) = h(x;r).$$

Now,

$$v(x;0,t) = 0 \cdot \overline{u}(x;0,t) = 0.$$

Therefore, $v(x;r,t)$ solves the one-dimensional wave equation on a half-line with Dirichlet boundary conditions, as claimed. □

Now we use this fact to construct the solution of (7.1). By d'Alembert's formula, we know that for $0 \leq r \leq t$, the solution $v(x;r,t)$ is given by

$$v(x;r,t) = \frac{1}{2}[g(x;r+t) - g(x;t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} h(x;y)\, dy.$$

Now

$$u(x,t) = \lim_{r\to 0^+} \overline{u}(x;r,t)$$

and

$$v(x;r,t) = r\overline{u}(x;r,t).$$

Therefore,

$$\begin{aligned}
u(x,t) &= \lim_{r\to 0^+} \frac{v(x;r,t)}{r} \\
&= \lim_{r\to 0^+} \left\{ \frac{1}{2r}[g(x;t+r) - g(x;t-r)] + \frac{1}{2r}\int_{-r+t}^{r+t} h(x;y)\, dy \right\} \\
&= \frac{d}{dt} g(x;t) + h(x;t).
\end{aligned}$$

Now

$$g(x;r) = r\overline{\phi}(x;r)$$

implies

$$g(x;t) = t\overline{\phi}(x;t) = t \fint_{\partial B(x,t)} \phi(y)\, dS(y).$$

Similarly,

$$h(x;t) = t\overline{\psi}(x;t) = t \fint_{\partial B(x,t)} \psi(y)\, dS(y).$$

Therefore, the solution of the wave equation in $\mathbb{R}^3$ (with $c = 1$) is given by

$$\boxed{u(x,t) = \frac{\partial}{\partial t}\left[t \fint_{\partial B(x,t)} \phi(y)\,dS(y)\right] + t \fint_{\partial B(x,t)} \psi(y)\,dS(y).}$$

If ϕ is smooth, the solution can be simplified further. In particular, for ϕ smooth, we have

$$\begin{aligned}
\frac{d}{dt} g(x;t) &= \frac{d}{dt}\left(t \fint_{\partial B(x,t)} \phi(y)\,dS(y)\right) \\
&= \frac{d}{dt}\left(t \fint_{\partial B(0,1)} \phi(x+tz)\,dS(z)\right) \\
&= \fint_{\partial B(0,1)} \phi(x+tz)\,dS(z) + t \int_{\partial B(0,1)} \nabla\phi(x+tz)\cdot z\,dS(z) \\
&= \fint_{\partial B(x,t)} \phi(y)\,dS(y) + t \fint_{\partial B(x,t)} \nabla\phi(y)\cdot\left(\frac{y-x}{t}\right)\,dS(y) \\
&= \fint_{\partial B(x,t)} \phi(y)\,dS(y) + \fint_{\partial B(x,t)} \nabla\phi(y)\cdot(y-x)\,dS(y).
\end{aligned}$$

And,

$$h(x;t) = t\overline{\psi}(x;t) = t \fint_{\partial B(x,t)} \psi(y)\,dS(y).$$

Therefore, we have

$$u(x,t) = \fint_{\partial B(x,t)} [\phi(y) + \nabla\phi(y)\cdot(y-x) + t\psi(y)]\,dS(y).$$

We note that in $\mathbb{R}^3$,

$$\fint_{\partial B(x,t)} = \frac{1}{n\alpha(n)t^{n-1}} \int_{\partial B(x,t)} = \frac{1}{4\pi t^2} \int_{\partial B(x,t)}.$$

Therefore, the solution of the IVP for the wave equation in $\mathbb{R}^3$ (with $c = 1$ and ϕ smooth) is given by

$$\boxed{u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y)\cdot(y-x) + t\psi(y)]\,dS(y).} \tag{7.2}$$

This is known as **Kirchoff's formula** for the solution of the initial value problem for the wave equation in $\mathbb{R}^3$.

Remark. Above we found the solution for the wave equation in $\mathbb{R}^3$ in the case when $c = 1$. If $c \neq 1$, we can simply use the above formula making a change of variables. In particular, consider the initial-value problem

$$\begin{cases} v_{tt} - c^2\Delta v = 0 & x \in \mathbb{R}^n \\ v(x,0) = \phi(x) \\ v_t(x,0) = \psi(x). \end{cases} \tag{7.3}$$

Suppose v is a solution of (7.3). Then define $u(x,t) \equiv v(x, \frac{1}{c}t)$. Then

$$u_{tt} - \Delta u = \frac{1}{c^2} v_{tt} - \Delta v = 0$$

implies u is a solution of

$$\begin{cases} u_{tt} - u_{xx} = 0 & x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \frac{1}{c}\psi(x). \end{cases}$$

Therefore, u is given by Kirchoff's formula above. Now by making the change of variables $\tilde{t} = \frac{1}{c}t$, we see that

$$v(x,\tilde{t}) = u(x, c\tilde{t}),$$

and we arrive at the solution for (7.3),

$$v(x,t) = \frac{1}{4\pi c^2 t^2} \int_{\partial B(x,ct)} [\phi(y) + \nabla\phi(y) \cdot (y-x) + t\psi(y)]\, dS(y).$$

7.2 Method of Descent

In this section, we use Kirchoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimensions. This technique is known as the **method of descent**. This technique can be used in general to find the solution of the wave equation in even dimensions, using the solution of the wave equation in odd dimensions.

Solution for $\mathbf{n} = 2$.

Suppose u is a solution of the initial value problem for the wave equation in two dimensions,

$$\begin{cases} u_{tt} - \Delta u = 0, & x \in \mathbb{R}^2, t \geq 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

We will find a solution in the 2-D case, by using the solution to the 3-D problem. Let $u(x_1, x_2, t)$ be the solution to the 2-D problem. Define

$$\widetilde{u}(x_1, x_2, x_3, t) \equiv u(x_1, x_2, t).$$

Therefore,

$$\begin{aligned} \widetilde{u}(x_1, x_2, x_3, 0) &\equiv u(x_1, x_2, 0) = \phi(x_1, x_2) \\ \widetilde{u}_t(x_1, x_2, x_3, 0) &\equiv u(x_1, x_2, 0) = \psi(x_1, x_2). \end{aligned}$$

Clearly, $\widetilde{u}(x_1, x_2, x_3, t)$ is a solution of the 3D wave equation with initial data $\phi(x_1, x_2)$ and $\psi(x_1, x_2)$,

$$\begin{cases} \widetilde{u}_{tt} - \widetilde{u}_{x_1x_1} - \widetilde{u}_{x_2x_2} - \widetilde{u}_{x_3x_3} = 0 \\ \widetilde{u}(x_1, x_2, x_3, 0) = \widetilde{\phi}(x_1, x_2, x_3) = \phi(x_1, x_2) \\ \widetilde{u}_t(x_1, x_2, x_3, 0) = \widetilde{\psi}(x_1, x_2, x_3) = \psi(x_1, x_2). \end{cases}$$

Now we can solve the 3D wave equation using Kirchoff's formula. In particular, our solution is given by

$$\widetilde{u}(x_1, x_2, 0, t) = ⨍_{\partial \overline{B}(\overline{x},t)} [\widetilde{\phi}(y) + \nabla \widetilde{\phi}(y) \cdot (y - x) + t\widetilde{\psi}(y)] \, dS(y)$$

where $\overline{B}(\overline{x}, t)$ is the ball of radius t in $\mathbb{R}^3$ about the point $\overline{x} = (x_1, x_2, 0)$. Now we note that

$$\begin{aligned} ⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y) \, dS(y) &= \frac{1}{4\pi t^2} \int_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y) \, dS(y) \\ &= \frac{1}{2\pi t^2} \int_{B(x,t)} \phi(y)(1 + |\nabla \gamma(y)|^2)^{1/2} \, dy \end{aligned}$$

where $B(x, t)$ is the ball in $\mathbb{R}^2$ of radius t about the point $x = (x_1, x_2)$ and $\gamma(y) = (t^2 - |y - x|^2)^{1/2}$. Therefore,

$$\nabla \gamma(y) = -\frac{y - x}{(t^2 - |y - x|^2)^{1/2}}$$

which implies

$$(1 + |\nabla \gamma(y)|^2)^{1/2} = \left(\frac{t^2}{t^2 - |y - x|^2} \right)^{1/2}.$$

Therefore,

$$⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y) \, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\phi(y)}{(t^2 - |y - x|^2)^{1/2}} \, dy.$$

Similarly,

$$⨍_{\partial \overline{B}(\overline{x},t)} t\widetilde{\psi}(y) \, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t^2\psi(y)}{(t^2 - |y - x|^2)^{1/2}} \, dy$$

and

$$⨍_{\partial \overline{B}(\overline{x},t)} \nabla \widetilde{\phi}(y) \cdot (y - x) \, dS(y) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\nabla \phi(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}} \, dy.$$

Therefore, the solution of the initial-value problem for the wave equation in $\mathbb{R}^2$ (with $c = 1$) is given by

$$\boxed{u(x, t) = \frac{1}{2\pi t^2} \int_{B(x,t)} \frac{t\phi(y) + t^2\psi(y) + t\nabla \phi(y) \cdot (y - x)}{(t^2 - |y - x|^2)^{1/2}} \, dy.} \tag{7.4}$$

Again, by making a change of variables, we see that the solution of the wave equation in two dimensions is given by

$$u(x, t) = \frac{1}{2\pi c^2 t^2} \int_{B(x,ct)} \frac{ct\phi(y) + ct^2\psi(y) + ct\nabla \phi(y) \cdot (y - x)}{(c^2t^2 - |y - x|^2)^{1/2}} \, dy.$$

7.3 Huygen's Principle

Note that for the initial-value problem for the wave equation in three dimensions, the value of the solution at any point $(x,t) \in \mathbb{R}^3 \times (0,\infty)$ depends only on the values of the initial data on the *surface of the ball* of radius ct about the point $x \in \mathbb{R}^3$; that is, on $\partial B(x,ct)$. That is to say, disturbances all travel at exactly speed c. This is known as **Huygens's principle**. In contrast, in two dimensions, the value of the solution u at the point (x,t) depends on the initial data *within* the ball of radius ct about the point $x \in \mathbb{R}^2$. Signals don't all travel at speed c. In fact, as we will see, for $n \geq 3$ and odd, Huygens's principle holds. That is, all signals travel at exactly speed c. In even dimensions, however, that is not the case.

7.4 Wave Equation in $\mathbb{R}^n$, $n > 3$

Ref: Evans, Sec. 2.4.1

Note: In this section, we assume c = 1. For c ≠ 1, we can make a change of variables to find the solution.

Odd dimensions.

For the case of odd dimensions, we use the method of spherical means as we did for the case of $n = 3$. Let $n = 2k+1$. Let $x \in \mathbb{R}^n$. Define

$$\begin{aligned} v(x;r,t) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{u}(x;r,t)) \\ g(x;r) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\phi}(x;r)) \\ h(x;r) &\equiv \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\psi}(x;r)). \end{aligned}$$

Notice that for $k = 1$, these definitions reduce to those functions introduced in the case $n = 3$.

First, we will show that $v(x;r,t)$ solves the wave equation on the half-line with Dirichlet boundary conditions.

Lemma 4. *For each integer $k \geq 1$, for each $x \in \mathbb{R}^n$, the function $v(x;r,t)$ defined above solves*

$$\begin{cases} v_{tt} - v_{rr} = 0 & r > 0 \\ v(x;r,0) = g(x;r) \\ v_t(x;r,0) = h(x;r) \\ v(x;0,t) = 0. \end{cases}$$

The proof relies on the following lemma.

Lemma 5. *Let $\phi : \mathbb{R} \to \mathbb{R}$ be C^{k+1}. Then for $k = 1, 2, \ldots$*

1.

$$\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} \left(r^{2k-1}\phi(r)\right) = \left(\frac{1}{r}\frac{d}{dr}\right)^{k} \left(r^{2k}\frac{d\phi}{dr}(r)\right)$$

2.

$$\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\phi(r)\right) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j\phi}{dr^j}(r)$$

where each β_j^k is independent of ϕ.

3.

$$\beta_0^k = 1\cdot 3\cdot 5\cdots(2k-1).$$

Proof. Use induction. □

Proof of Lemma 4.

$$\begin{aligned}
v_{rr} &= \partial_r^2\left[\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}(x;r,t)\right)\right] \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left(r^{2k}\overline{u}_r(x;r,t)\right) \qquad \text{by Lemma 5} \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}\frac{d}{dr}\right)\left(r^{2k}\overline{u}_r(x;r,t)\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(\frac{1}{r}[2kr^{2k-1}\overline{u}_r + r^{2k}\overline{u}_{rr}]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left[\frac{2k}{r}\overline{u}_r + \overline{u}_{rr}\right]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\left[\frac{n-1}{r}\overline{u}_r + \overline{u}_{rr}\right]\right) \\
&= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}_{tt}\right) \\
&= \partial_t^2\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left(r^{2k-1}\overline{u}_{tt}\right) \\
&= v_{tt}
\end{aligned}$$

Clearly, $v(x;r,0) = g(x;r)$, $v_t(x;r,0) = h(x;r)$ and $v(x;0,t) = 0$. Therefore, the lemma is proved. □

Now $v(x;r,t)$ is a solution of the one-dimensional wave equation on the half-line with Dirichlet boundary condition implies for $0 \le r \le t$, the solution is given by

$$v(x;r,t) = \frac{1}{2}[g(x;r+t) - g(x;t-r)] + \frac{1}{2}\int_{t-r}^{t+r} h(x;y)\,dy.$$

Recall:

$$u(x,t) = \lim_{r\to 0}\overline{u}(x;r,t).$$

Now

$$\begin{aligned} v(x;r,t) &= \left(\frac{1}{r}\frac{d}{dr}\right)^{k-1} \left(r^{2k-1}\overline{u}(x;r,t)\right) \\ &= \sum_{j=0}^{k-1} \beta_j^k r^{j+1} \frac{\partial^j}{\partial r^j} \overline{u}(x;r,t) \\ &= \beta_0^k r \overline{u}(x;r,t) + \beta_1^k r^2 \overline{u}_r(x;r,t) + \ldots + \beta_{k-1}^k r^k \frac{\partial^{k-1}}{\partial r^{k-1}} \overline{u}(x;r,t). \end{aligned}$$

Therefore,

$$\beta_0^k r \overline{u}(x;r,t) = v(x;r,t) - \beta_1^k r^2 \overline{u}_r(x;r,t) - \ldots - \beta_{k-1}^k r^k \frac{\partial^{k-1}}{\partial r^{k-1}} \overline{u}(x;r,t),$$

which implies

$$\overline{u}(x;r,t) = \frac{v(x;r,t)}{\beta_0^k r} - \frac{\beta_1^k}{\beta_0^k} r \overline{u}_r(x;r,t) - \ldots - \frac{\beta_{k-1}^k}{\beta_0^k} r^{k-1} \frac{\partial^{k-1}}{\partial r^{k-1}} \overline{u}(x;r,t).$$

Therefore,

$$\begin{aligned} u(x,t) &= \lim_{r\to 0} \left[\frac{v(x;r,t)}{\beta_0^k r} - \frac{\beta_1^k}{\beta_0^k} r \overline{u}_r(x;r,t) - \ldots - \frac{\beta_{k-1}^k}{\beta_0^k} r^{k-1} \frac{\partial^{k-1}}{\partial r^{k-1}} \overline{u}(x;r,t) \right] \\ &= \lim_{r\to 0} \frac{v(x;r,t)}{\beta_0^k r} \\ &= \lim_{r\to 0} \frac{1}{\beta_0^k} \left[\frac{g(x;t+r) - g(x;t-r)}{2r} + \frac{1}{2r} \int_{t-r}^{t+r} h(x;y)\, dy \right] \\ &= \frac{1}{\beta_0^k} \left[\partial_t g(x;t) + h(x;t) \right] \end{aligned}$$

where $\beta_0^k = 1 \cdot 3 \cdot 5 \cdots (2k-1)$. Recall

$$g(x;r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\phi}(x;r)).$$

Now $n = 2k+1$ implies $k = (n-1)/2$, and, therefore,

$$g(x;t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B(x,t)} \phi(y)\, dS(y)\right).$$

And,

$$h(x;r) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} (r^{2k-1}\overline{\psi}(x;r)).$$

Therefore,

$$h(x;t) = \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}} \left(t^{n-2} \fint_{\partial B(x,t)} \psi(y)\, dS(y)\right).$$

Therefore,

$$u(x,t) = \frac{1}{\gamma_n}[\partial_t g(x;t) + h(x;t)]$$

implies

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{\gamma_n}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}⨍_{\partial B(x,t)} \phi(y)\,dS(y)\right) \\ + \frac{1}{\gamma_n}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-3}{2}}\left(t^{n-2}⨍_{\partial B(x,t)} \psi(y)\,dS(y)\right) \end{aligned}}$$

where $\gamma_n = 1 \cdot 3 \cdot 5 \cdots (n-2)$.

Even dimensions.

As in the case of $n = 2$ dimensions, we use the method of descent. In particular, suppose $u(x_1, \ldots, x_n, t)$ is a solution of the wave equation in $\mathbb{R}^n$ with initial data $u(x_1, \ldots, x_n, 0) = \phi(x_1, \ldots, x_n)$ and $u_t(x_1, \ldots, x_n, 0) = \psi(x_1, \ldots, x_n)$. Then define

$$\begin{aligned} \widetilde{u}(x_1, \ldots, x_{n+1}, t) &\equiv u(x_1, \ldots, x_n, t) \\ \widetilde{\phi}(x_1, \ldots, x_{n+1}) &\equiv \phi(x_1, \ldots, x_n) \\ \widetilde{\psi}(x_1, \ldots, x_{n+1}) &\equiv \psi(x_1, \ldots, x_n). \end{aligned}$$

Therefore, $\widetilde{u}$ is a solution of the wave equation in $\mathbb{R}^{n+1}$, where now $n+1$ is odd. Therefore, from the formula above for the case when the dimension is odd, our solution at the point $(\overline{x}, t) = (x_1, \ldots, x_n, 0, t)$ is given by

$$\begin{aligned} \widetilde{u}(\overline{x},t) = \frac{1}{\gamma_{n+1}}\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1}⨍_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y)\right) \\ + \frac{1}{\gamma_{n+1}}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^{n-1}\oint_{\partial\overline{B}(\overline{x},t)} \widetilde{\psi}(y)\,dS(y)\right) \end{aligned}$$

where $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdots (n-1)$, and where $\overline{B}(\overline{x}, t)$ is the ball in $\mathbb{R}^{n+1}$ of radius t about the point $\overline{x} = (x_1, \ldots, x_n, 0)$.

Now,

$$⨍_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y) = \frac{1}{(n+1)\alpha(n+1)t^n}\int_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y).$$

But, notice $\partial\overline{B}(\overline{x}, t) \cap \{y_{n+1} \geq 0\}$ is the graph of the function $\gamma(y) \equiv (t^2 - |y-x|^2)^{1/2}$. And, similarly, $\partial\overline{B}(\overline{x}, t) \cap \{y_{n+1} \leq 0\}$ is the graph of $-\gamma$. Therefore,

$$\frac{1}{(n+1)\alpha(n+1)t^n}\int_{\partial\overline{B}(\overline{x},t)} \widetilde{\phi}(y)\,dS(y) = \frac{2}{(n+1)\alpha(n+1)t^n}\int_{B(x,t)} \phi(y)(1+|\nabla\gamma(y)|^2)^{1/2}\,dy$$

Now

$$(1+|\nabla\gamma(y)|^2)^{1/2} = t(t^2 - |y-x|^2)^{-1/2}.$$

Therefore,

$$\begin{aligned}
⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y) &= \frac{2}{(n+1)\alpha(n+1)t^n} \int_{B(x,t)} \frac{t\phi(y)}{(t^2-|y-x|^2)^{1/2}}\, dy \\
&= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)\alpha(n)t^n} \int_{B(x,t)} \frac{\phi(y)}{(t^2-|y-x|^2)^{1/2}}\, dy \\
&= \frac{2t\alpha(n)}{(n+1)\alpha(n+1)} ⨍_{B(x,t)} \frac{\phi(y)}{(t^2-|y-x|^2)^{1/2}}\, dy.
\end{aligned}$$

Therefore, our solution formula is given by

$$\begin{aligned}
u(x,t) &= \frac{1}{\gamma_{n+1}} \left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\phi}(y)\, dS(y)\right) \\
&\quad + \frac{1}{\gamma_{n+1}} \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^{n-1} ⨍_{\partial \overline{B}(\overline{x},t)} \widetilde{\psi}(y)\, dS(y)\right) \\
&= \frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} \left[\left(\frac{\partial}{\partial t}\right) \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n ⨍_{\partial B(x,t)} \frac{\phi(y)}{(t^2-|y-x|^2)^{1/2}}\, dy\right) \right. \\
&\qquad \left. + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}} \left(t^n ⨍_{\partial B(x,t)} \frac{\psi(y)}{(t^2-|y-x|^2)^{1/2}}\, dy\right) \right].
\end{aligned}$$

Now $\gamma_{n+1} = 1 \cdot 3 \cdot 5 \cdots (n-1)$ and

$$\alpha(n) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n+2}{2}\right)},$$

where $\Gamma(n)$ is the gamma function,

$$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\, dx.$$

Therefore,

$$\begin{aligned}
\frac{1}{\gamma_{n+1}} \cdot \frac{2\alpha(n)}{(n+1)\alpha(n+1)} &= \frac{1}{1\cdot 3\cdot 5\cdots(n-1)} \cdot \frac{2\frac{\pi^{n/2}}{\Gamma(\frac{n+2}{2})}}{(n+1)\frac{\pi^{(n+1)/2}}{\Gamma(\frac{n+3}{2})}} \\
&= \frac{1}{1\cdot 3\cdot 5\cdots(n+1)} \cdot \frac{1}{\pi^{1/2}} \cdot \frac{\Gamma(\frac{n+3}{2})}{\Gamma(\frac{n+2}{2})}.
\end{aligned}$$

Using properties of the gamma function, namely that

$$\Gamma(m+1) = m\Gamma(m)$$

and

$$\Gamma(1/2) = \pi^{1/2},$$

we can conclude that

$$\Gamma\left(\frac{n+3}{2}\right) = \left(\frac{n+1}{2}\right)\cdot\left(\frac{n-1}{2}\right)\cdots\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)$$

and

$$\Gamma\left(\frac{n+2}{2}\right) = \left(\frac{n}{2}\right)\cdot\left(\frac{n-2}{2}\right)\cdots\left(\frac{2}{2}\right).$$

And, therefore,

$$\frac{1}{\gamma_{n+1}}\cdot\frac{2\alpha(n)}{(n+1)\alpha(n+1)} = \frac{1}{2\cdot 4\cdots(n-2)\cdot n}$$

Therefore, the solution of the wave equation in even dimensions is given by

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{\gamma_n}\Bigg[&\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{\phi(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right) \\ &+\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{n-2}{2}}\left(t^n \fint_{B(x,t)} \frac{\psi(y)}{(t^2-|y-x|^2)^{1/2}}\,dy\right)\Bigg]\end{aligned}}$$

where $\gamma_n \equiv 2\cdot 4\cdots(n-2)\cdot n$.

7.5 Wave Equation in $\mathbb{R}^n$ with a source.

In this section, we consider the inhomogeneous wave equation in $\mathbb{R}^n$. First, recall Duhamel's Principle. If $S(t)$ is the solution operator for the first-order initial-value problem

$$\begin{cases} U_t + AU = 0 \\ U(0) = \Phi, \end{cases}$$

then the solution of the inhomogeneous problem

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi \end{cases}$$

"should" be given by

$$U(t) = S(t)\Phi + \int_0^t S(t-s)F(s)\,ds.$$

Now consider the initial-value problem for the wave equation in $\mathbb{R}^n$,

$$\begin{cases} u_{tt} - \Delta u = f(x,t) \qquad x \in \mathbb{R}^n \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases} \tag{7.5}$$

Introducing a new function $v = u_t$, we can rewrite this equation as

$$\begin{cases} \begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ \Delta & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f \end{bmatrix} & x \in \mathbb{R}^n \\ \begin{bmatrix} u(x,0) \\ v(x,0) \end{bmatrix} = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}. \end{cases} \tag{7.6}$$

or

$$\begin{cases} U_t + AU = F \\ U(x,0) = \Phi(x) \end{cases}$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -\Delta & 0 \end{bmatrix}$$
$$F = \begin{bmatrix} 0 \\ f \end{bmatrix} \qquad \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}.$$

Now in order to solve (7.5), we look for the solution operator $S(t)$ associated with the first-order system (7.6).

First, consider the case $n = 3$. In three dimensions, we can find the solution operator $S(t)$ by using Kirchoff's formula. Recall that the solution of the initial-value problem for the homogeneous wave equation in three dimensions (with $c = 1$) is given by

$$u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y) \cdot (y - x) + t\psi(y)]\, dS(y),$$

which implies the solution operator $S(t)$ associated with (7.6) is given by

$$S(t)\Phi = S(t) \begin{bmatrix} \phi \\ \psi \end{bmatrix} = \begin{bmatrix} \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y) \cdot (y - x) + t\psi(y)]\, dS(y) \\ \partial_t \left(\frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y) \cdot (y - x) + t\psi(y)]\, dS(y) \right) \end{bmatrix}.$$

Therefore,

$$S(t-s)F(s) = S(t-s) \begin{bmatrix} 0 \\ f(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y,s)\, dS(y) \\ \partial_t \left(\frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y,s)\, dS(y) \right) \end{bmatrix}.$$

Now using the fact that the solution of (7.6) is given by

$$\begin{bmatrix} u(x,t) \\ v(x,t) \end{bmatrix} = U(x,t) = S(t)\Phi(x) + \int_0^t S(t-s)F(x,s)\, ds,$$

we see that the solution of (7.5) is given by the first component of U. Therefore, the solution of the initial-value problem for the inhomogeneous wave equation in three dimensions (with $c = 1$) (7.5) is given by

$$\boxed{\begin{aligned} u(x,t) = \frac{1}{4\pi t^2} \int_{\partial B(x,t)} [\phi(y) + \nabla\phi(y) \cdot (y - x) + t\psi(y)]\, dS(y) \\ + \int_0^t \frac{1}{4\pi(t-s)} \int_{\partial B(x,t-s)} f(y,s)\, dS(y)\, ds. \end{aligned}}$$

Similarly, in two dimensions, the first component of the solution operator is given by

$$S_1(t)\Phi = S_1(t)\begin{bmatrix}\phi\\ \psi\end{bmatrix} = \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\phi(y) + t^2\psi(y) + t\nabla\phi(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy.$$

Therefore, the solution of the initial-value problem for the inhomogeneous wave equation in two dimensions (with $c = 1$) (7.5) is given by

$$\boxed{\begin{aligned} u(x,t) = {} & \frac{1}{2\pi t^2}\int_{B(x,t)} \frac{t\phi(y) + t^2\psi(y) + t\nabla\phi(y)\cdot(y-x)}{(t^2 - |y-x|^2)^{1/2}}\,dy \\ & + \int_0^t \frac{1}{2\pi(t-s)^2}\int_{B(x,t-s)} \frac{(t-s)^2 f(y,s)}{((t-s)^2 - |y-x|^2)^{1/2}}\,dy\,ds. \end{aligned}}$$

7.6 Wave Equation on a Bounded Domain in $\mathbb{R}^n$.

In this section, we consider the initial-value problem for the wave equation on a bounded domain $\Omega \subset \mathbb{R}^n$,

$$\begin{cases} u_{tt} - c^2\Delta u = 0, & x \in \Omega \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u \text{ satisfies certain boundary conditions on } \partial\Omega, \end{cases}$$

As before, we look for a solution using separation of variables. In particular, we look for a solution of the form $u(x,t) = X(x)T(t)$. Substituting a function of this form into our PDE, we arrive at the equation

$$T''X - c^2 T\Delta X = 0.$$

This equation implies the functions T and X satisfy the following equation for some scalar λ,

$$-\frac{T''}{c^2 T} = -\frac{\Delta X}{X} = \lambda.$$

Consequently, we are lead to the following eigenvalue problem

$$\begin{cases} -\Delta X = \lambda X, & x \in \Omega \\ X \text{ satisfies certain boundary conditions on } \partial\Omega. \end{cases}$$

Suppose we find eigenvalues λ_n with corresponding eigenfunctions $X_n(x)$. Then for each n, we just need to solve

$$T_n''(t) + c^2\lambda_n T_n(t) = 0.$$

If λ_n is positive, this means

$$T_n(t) = A_n\cos(\sqrt{\lambda_n}ct) + B_n\sin(\sqrt{\lambda_n}ct).$$

If $\lambda_n = 0$, this means

$$T_n(t) = A_n + B_n t.$$

If λ_n is negative, this means

$$T_n(t) = A_n \cosh(\sqrt{-\lambda_n} ct) + B_n \sinh(\sqrt{-\lambda_n} ct).$$

Then defining the function

$$u(x,t) = \sum_n T_n(t) X_n(x),$$

for X_n, T_n as defined above for any choice of constants A_n, B_n, we have found a solution of the wave equation on the bounded domain $\Omega \subset \mathbb{R}^n$, which satisfies our boundary conditions.

Now in order for our initial conditions to be satisfied, that is, $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, we need to choose constants A_n, B_n such that

$$u(x,0) = \sum_n A_n X_n(x) = \phi(x)$$

and

$$u_t(x,0) = \sum_{\lambda_n \neq 0} c\sqrt{|\lambda_n|} B_n X_n(x) + \sum_{\lambda_n = 0} B_n X_n(x) = \psi(x).$$

If our eigenfunctions are orthogonal, then we can find coefficients A_n, B_n satisfying the above equations, by multiplying these equations by X_m for a fixed m and integrating over Ω. Doing so, we see that our coefficients A_n are given by

$$A_n = \frac{\langle X_n, \phi \rangle}{\langle X_n, X_n \rangle} = \frac{\int_\Omega X_n(x)\phi(x)\,dx}{\int_\Omega X_n^2(x)\,dx},$$

and

$$\begin{aligned} c\sqrt{|\lambda_n|} B_n &= \frac{\langle X_n, \psi \rangle}{\langle X_n, X_n \rangle} = \frac{\int_\Omega X_n(x)\psi(x)\,dx}{\int_\Omega X_n^2(x)\,dx} \qquad \text{for } \lambda_n \neq 0 \\ B_n &= \frac{\langle X_n, \psi \rangle}{\langle X_n, X_n \rangle} = \frac{\int_\Omega X_n(x)\psi(x)\,dx}{\int_\Omega X_n^2(x)\,dx} \qquad \text{for } \lambda_n = 0. \end{aligned}$$