Math 220A - Fall 2002
Homework 2
Due Friday, Oct. 11, 2002

1. Solve
$$\begin{cases} u_x^2 u_t - 1 = 0 \\ u(x,0) = x. \end{cases}$$

2. Solve
$$\begin{cases} u_t + u_x^2 + u = 0 \\ u(x,0) = x. \end{cases}$$

3. Assume $(\vec{x}(\vec{r},s), z(\vec{r},s), \vec{p}(\vec{r},s))$ is the solution of the characteristic ODEs for the fully nonlinear first-order equation
$$\begin{cases} F(\vec{x}, u, Du) = 0 \\ u|_\Gamma = \phi \end{cases}$$
which satisfies the initial condition $(\vec{x}(\vec{r},0), z(\vec{r},0), \vec{p}(\vec{r},0)) = (\Gamma(\vec{r}), \phi(\vec{r}), \Psi(\vec{r}))$, where (Γ, ϕ, Ψ) is admissible initial data. Show that
$$\frac{d}{ds} F(\vec{x}, z, \vec{p}) = 0.$$
Note: This result proves part of the local existence theorem.

4. Consider the initial-value problem
$$(*) \begin{cases} u_t + a u_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$
We say u is a weak solution of (*) if u satisfies
$$\int_0^\infty \int_{-\infty}^\infty u[v_t + a v_x]\,dx\,dt + \int_{-\infty}^\infty \phi(x)v(x)\,dx = 0$$
for all $v \in C^\infty(\mathbb{R}^n \times [0,\infty))$ with compact support. Assume that ϕ is a piecewise C^1 function. Show that $u(x,t) = \phi(x - at)$ is a weak solution of (*).

5. Consider the initial-value problem
$$(*) \begin{cases} [g(u)]_t + [f(u)]_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$
We say u is a weak solution of (*) if u satisfies
$$\int_0^\infty \int_{-\infty}^\infty g(u)v_t + f(u)v_x\,dx\,dt + \int_{-\infty}^\infty g(\phi(x))v(x,0)\,dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support. Suppose u is a weak solution of (*) such that u has a jump discontinuity across the curve $x = \xi(t)$, but u is smooth on either side of the curve $x = \xi(t)$. Let $u^-(x,t)$ be the value of u to the left of the curve and $u^+(x,t)$ be the value of u to the right of the curve. Prove that u must satisfy the condition

$$\frac{[f(u)]}{[g(u)]} = \xi'(t)$$

across the curve of discontinuity, where

$$\begin{aligned} [f(u)] &= f(u^-) - f(u^+) \\ [g(u)] &= g(u^-) - g(u^+). \end{aligned}$$