Math 220a - Fall 2002
Homework 4
Due Friday, Oct. 25, 2002

1. Classify the following equations as elliptic, parabolic, or hyperbolic.

   (a) $2u_{xx} + 2u_{xy} + 2u_{xz} + 3u_{yy} - 4u_{yz} + 3u_{zz} = 0$

   (b) $2u_{xz} + u_{yy} = 0$

   (c) $7u_{xx} - 10u_{xy} - 22u_{yz} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0$

2. Show that every elliptic equation of the form

$$au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x, y),$$

   where $b^2 - 4ac < 0$ can be brought into the form

$$\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + k\tilde{u} = F(\xi, \eta)$$

   through a change of variables. In particular, first give an appropriate linear change of variables to show the equation can be written in the form

$$u_{\xi\xi} + u_{\eta\eta} + \alpha u_\xi + \beta u_\eta + \gamma u = h(\xi, \eta)$$

   for some constants α, β, γ. Then introduce a change of variables for the dependent variable u to eliminate the first derivative terms, to show that the equation can be written in the form

$$\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + k\tilde{u} = F(\xi, \eta).$$

3. Reduce the following second-order equation to a system of first-order equations

$$u_{tt} - 4u_{xt} - 5u_{xx} = 0.$$

   Then use the method of characteristics to derive the general solution.

4. Consider the IVP

$$(*) \begin{cases} u_{tt} + u_{xt} - 12u_{xx} = 0 \\ u(x, 0) = \phi(x) \\ u_t(x, 0) = \psi(x) \end{cases}$$

   (a) Make a change of variables to reduce the PDE to canonical form

$$u_{\xi\xi} - u_{\eta\eta} = 0.$$

   Write the general solution of

$$u_{tt} + u_{xt} - 12u_{xx} = 0.$$

(b) Solve the IVP (*).

5. We say u is a weak solution of the wave equation,

$$\begin{cases} u_{tt} - u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

if

$$\int_0^\infty \int_{-\infty}^\infty u[v_{tt} - v_{xx}]\,dx\,dt + \int_{-\infty}^\infty \phi(x)v_t(x,0)\,dx - \int_{-\infty}^\infty \psi(x)v(x,0)\,dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0,\infty))$ with compact support. Let f be a piecewise continuous function with a jump at y_0. Show that $u(x,t) = f(x+t)$ is a weak solution of the wave equation. (*Note:* Similarly it can be shown that a piecewise continuous function of the form $g(x-t)$ is also a weak solution.)