Math 220A - Fall 2002
Homework 5
Due Friday, Nov. 1, 2002

1. Consider the initial-value problem for the hyperbolic equation

$$\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

   Use energy methods to show that the domain of dependence of the solution u at the point (x_0, t_0) is the cone $\{(x,t) \in \mathbb{R}^2 : t \geq 0, x_0 - 5(t_0 - t) \leq x \leq x_0 + 4(t_0 - t)\}$.

2. Use energy methods to prove uniqueness of solutions to

$$\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = f(x,t) & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

   assuming that ϕ and ψ have compact support.

3. Consider the initial-value problem for the following hyperbolic equation,

$$\begin{cases} ru_{tt} - \nabla \cdot (p\nabla u) + qu = F & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

   where $r(x), p(x)$ are positive and $q(x)$ is non-negative. Use energy methods to prove uniqueness of solutions to this problem.

4. Use Duhamel's principle to derive formulas for the solutions of the following initial value problems.

   (a)

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

      i. First find the solution operator $S(t)$ associated with the homogeneous equation.

      ii. Use $S(t)$ to derive the solution of the inhomogeneous equation.

   (b)

$$(*) \begin{cases} u_{tt} + u_{xt} - 20u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

i. Write the equation as a system

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi \end{cases}$$

ii. Find the solution operator $S(t)$ associated with the homogeneous system

$$\begin{cases} U_t + AU = 0 \\ U(0) = \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \end{cases}$$

iii. Use the solution operator $S(t)$ to find the solution of the inhomogeneous system, and use this to find the solution of (*).

5. Use Green's Theorem to derive the solution of the inhomogeneous wave equation on the half-line,

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < \infty \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u_t(x,0) = \psi(x) & 0 < x < \infty \\ u(0,t) = h(t), & \end{cases}$$

where we assume $\phi(0) = \psi(0) = h(0) = 0$.