1. Consider the initial-value problem for the hyperbolic equation
\[
\begin{align*}
&u_{tt} + u_{xt} - 20u_{xx} = 0 & -\infty < x < \infty, t > 0 \\
&u(x, 0) = \phi(x) \\
&u_t(x, 0) = \psi(x).
\end{align*}
\]
Use energy methods to show that the domain of dependence of the solution $u$ at the point $(x_0, t_0)$ is the cone \{(x, t) \in \mathbb{R}^2 : t \geq 0, x_0 - 5(t_0 - t) \leq x \leq x_0 + 4(t_0 - t)\}.

2. Use energy methods to prove uniqueness of solutions to
\[
\begin{align*}
&u_{tt} + u_{xt} - 20u_{xx} = f(x, t) & -\infty < x < \infty, t > 0 \\
&u(x, 0) = \phi(x) \\
&u_t(x, 0) = \psi(x)
\end{align*}
\]
assuming that $\phi$ and $\psi$ have compact support.

3. Consider the initial-value problem for the following hyperbolic equation,
\[
\begin{align*}
&r u_{tt} - \nabla \cdot (p \nabla u) + qu = F & x \in \mathbb{R}^n, t > 0 \\
&u(x, 0) = \phi(x) \\
&u_t(x, 0) = \psi(x)
\end{align*}
\]
where $r(x), p(x)$ are positive and $q(x)$ is non-negative. Use energy methods to prove uniqueness of solutions to this problem.

4. Use Duhamel’s principle to derive formulas for the solutions of the following initial value problems.
(a)
\[
\begin{align*}
&u_t + au_x = f(x, t) \\
&u(x, 0) = \phi(x)
\end{align*}
\]
i. First find the solution operator $S(t)$ associated with the homogeneous equation.
ii. Use $S(t)$ to derive the solution of the inhomogeneous equation.
(b)
\[
\begin{align*}
&u_{tt} + u_{xt} - 20u_{xx} = f(x, t) \\
&(*) & u(x, 0) = \phi(x) \\
&u_t(x, 0) = \psi(x)
\end{align*}
\]
i. Write the equation as a system

\[
\begin{aligned}
&U_t + AU = F \\
&U(0) = \Phi
\end{aligned}
\]

ii. Find the solution operator \( S(t) \) associated with the homogeneous system

\[
\begin{aligned}
&U_t + AU = 0 \\
&U(0) = \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}.
\end{aligned}
\]

iii. Use the solution operator \( S(t) \) to find the solution of the inhomogeneous system, and use this to find the solution of (*)

5. Use Green’s Theorem to derive the solution of the inhomogeneous wave equation on the half-line,

\[
\begin{aligned}
&u_{tt} - c^2 u_{xx} = f(x, t) \quad 0 < x < \infty \\
u(x, 0) = \phi(x) \quad 0 < x < \infty \\
u_t(x, 0) = \psi(x) \quad 0 < x < \infty \\
u(0, t) = h(t),
\end{aligned}
\]

where we assume \( \phi(0) = \psi(0) = h(0) = 0 \).