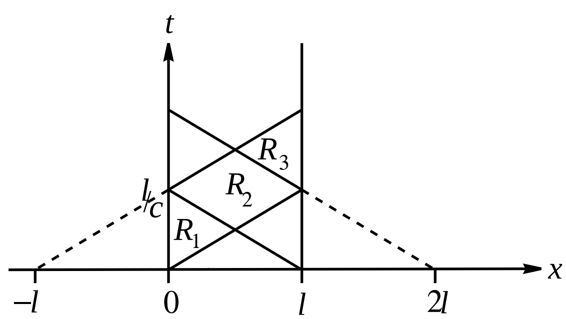
Math 220A - Fall 2002
Homework 6
Due Friday, November 15, 2002

1. Use the method of reflection to solve the initial-boundary value problem on the interval $0 < x < l$ with Dirichlet boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = x & 0 < x < l \\ u(0,t) = 0 = u(l,t). & \end{cases}$$

   In particular, calculate the explicit solution of u in regions R_1, R_2, R_3 shown below.



2. Do the same thing as in problem 1, except now for the *Neumann* boundary conditions. That is, use the method of reflection to solve the initial-boundary value problem on the interval $0 < x < l$ with Neumann boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = x & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). & \end{cases}$$

   Write the explicit solution in the same three regions as shown in problem 1.

3. Use Duhamel's principle to find the solution of the *inhomogeneous* wave equation on the half-line with Neumann boundary conditions

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & 0 < x < \infty \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u_t(x,0) = \psi(x) & 0 < x < \infty \\ u_x(0,t) = 0. & \end{cases}$$

   In particular, introducing a new function $v = u_t$, rewrite the equation as the system

$$\begin{cases} U_t + AU = F & 0 < x < \infty \\ U(x,0) = \Phi(x) & 0 < x < \infty \\ U_x(0,t) = \begin{bmatrix} u_x(0,t) \\ v_x(0,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} & \end{cases}$$

where

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \qquad A = \begin{bmatrix} 0 & -1 \\ -c^2\partial_x^2 & 0 \end{bmatrix}$$

$$F = \begin{bmatrix} 0 \\ f \end{bmatrix} \qquad \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}.$$

(a) Find the solution operator $S(t)$ associated with the homogeneous system

$$\begin{cases} U_t + AU = 0 & 0 < x < \infty \\ U(x,0) = \Phi(x) & 0 < x < \infty \\ U_x(0,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{cases}$$

(b) Use $S(t)$ to construct a solution of the inhomogeneous system.

(c) Use the solution of the inhomogeneous system to solve the inhomogeneous wave equation on the half-line with Neumann boundary conditions.

4. Use separation of variables to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l, t > 0 \\ u(x,0) = x(x-l)^2 & 0 < x < l \\ u_t(x,0) = 0 & 0 < x < l \\ u(0,t) = 0,\ u_x(l,t) = 0. \end{cases}$$

5. Consider the eigenvalue problem,

$$\begin{cases} -X'' = \lambda X & 0 < x < 1 \\ X'(0) + aX(0) = 0 \\ X(1) = 0. \end{cases}$$

(a) Find all positive eigenvalues. Show graphically that there is an infinite sequence of positive eigenvalues $\lambda_n \to +\infty$.

(b) For what values of a (if any) is zero an eigenvalue?

(c) Show that if $a \leq 1$ there are no negative eigenvalues, while if $a > 1$ there is one negative eigenvalue.

6. Use separation of variables to solve

$$\begin{cases} u_{tt} - c^2 u_{xx} + \gamma^2 u = 0, & 0 < x < l, t \geq 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \\ u(0,t) = 0 = u(l,t), & t \geq 0 \end{cases}$$

where $\gamma > 0$.