1. Use the method of reflection to solve the initial-boundary value problem on the interval \(0 < x < l\) with Dirichlet boundary conditions,

\[
\begin{align*}
&u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < l \\
u(x, 0) = 0 \quad 0 < x < l \\
u_t(x, 0) = x \quad 0 < x < l \\
u(0, t) = 0 = u(l, t).
\end{align*}
\]

In particular, calculate the explicit solution of \(u\) in regions \(R_1, R_2, R_3\) shown below.

2. Do the same thing as in problem 1, except now for the Neumann boundary conditions. That is, use the method of reflection to solve the initial-boundary value problem on the interval \(0 < x < l\) with Neumann boundary conditions,

\[
\begin{align*}
&u_{tt} - c^2 u_{xx} = 0 \quad 0 < x < l \\
u(x, 0) = 0 \quad 0 < x < l \\
u_t(x, 0) = x \quad 0 < x < l \\
u_x(0, t) = 0 = u_x(l, t).
\end{align*}
\]

Write the explicit solution in the same three regions as shown in problem 1.

3. Use Duhamel’s principle to find the solution of the inhomogeneous wave equation on the half-line with Neumann boundary conditions

\[
\begin{align*}
&u_{tt} - c^2 u_{xx} = f(x, t), \quad 0 < x < \infty \\
u(x, 0) = \phi(x) \quad 0 < x < \infty \\
u_t(x, 0) = \psi(x) \quad 0 < x < \infty \\
u_x(0, t) = 0.
\end{align*}
\]

In particular, introducing a new function \(v = u_t\), rewrite the equation as the system

\[
\begin{align*}
&U_t + AU = F \quad 0 < x < \infty \\
U(x, 0) = \Phi(x) \quad 0 < x < \infty \\
U_x(0, t) = \begin{bmatrix} u_x(0, t) \\ v_x(0, t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\end{align*}
\]
where
\[
U = \begin{bmatrix} u \\ v \end{bmatrix}, \quad A = \begin{bmatrix} 0 & -1 \\ -c^2 \partial_x^2 & 0 \end{bmatrix},
\]
\[
F = \begin{bmatrix} 0 \\ f \end{bmatrix}, \quad \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}.
\]

(a) Find the solution operator \( S(t) \) associated with the homogeneous system
\[
\begin{cases}
U_t + AU = 0 & 0 < x < \infty \\
U(x, 0) = \Phi(x) & 0 < x < \infty \\
U_x(0, t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{cases}
\]

(b) Use \( S(t) \) to construct a solution of the inhomogeneous system.

(c) Use the solution of the inhomogeneous system to solve the inhomogeneous wave equation on the half-line with Neumann boundary conditions.

4. Use separation of variables to solve
\[
\begin{cases}
\begin{aligned}
&u_{tt} - c^2 u_{xx} = 0 & 0 < x < l, t > 0 \\
u(x, 0) = x(x - l)^2 & 0 < x < l \\
u_t(x, 0) = 0 & 0 < x < l \\
u(0, t) = 0, & u_x(l, t) = 0.
\end{aligned}
\end{cases}
\]

5. Consider the eigenvalue problem,
\[
\begin{cases}
&-X'' = \lambda X & 0 < x < 1 \\
&X'(0) + aX(0) = 0 \\
&X(1) = 0.
\end{cases}
\]

(a) Find all positive eigenvalues. Show graphically that there is an infinite sequence of positive eigenvalues \( \lambda_n \to +\infty \).

(b) For what values of \( a \) (if any) is zero an eigenvalue?

(c) Show that if \( a \leq 1 \) there are no negative eigenvalues, while if \( a > 1 \) there is one negative eigenvalue.

6. Use separation of variables to solve
\[
\begin{cases}
\begin{aligned}
&u_{tt} - c^2 u_{xx} + \gamma^2 u = 0, & 0 < x < l, t \geq 0 \\
u(x, 0) = \phi(x) \\
u_t(x, 0) = \psi(x) \\
u(0, t) = 0 = u(l, t), & t \geq 0
\end{aligned}
\end{cases}
\]

where \( \gamma > 0 \).