1. Solve
\[
\begin{aligned}
&u_xu_t - 1 = 0 \\
&u(x, 0) = x.
\end{aligned}
\]
Answer: Let
\[
F(p, q, z, x, t) = p^2q - 1.
\]
The set of characteristic equations are given by
\[
\begin{aligned}
\frac{dx}{ds} &= 2pq \\
\frac{dy}{ds} &= p^2 \\
\frac{dz}{ds} &= 3 \\
\frac{dp}{ds} &= 0 \\
\frac{dq}{ds} &= 0
\end{aligned}
\]
\[x(r, 0) = r, \quad \quad t(r, 0) = 0, \quad \quad z(r, 0) = r, \quad \quad p(r, 0) = \psi_1(r), \quad \quad q(r, 0) = \psi_2(r),\]
where \(\psi_1, \psi_2\) satisfy
\[
\begin{aligned}
\phi'(r) &= \psi_1(r) \\
\psi_1^2\psi_2 - 1 &= 0.
\end{aligned}
\]
Therefore,
\[
\psi_1(r) = 1 = \psi_2(r).
\]
Solving this system of ODEs, we have
\[
\begin{aligned}
p &= 1 \\
q &= 1 \\
x &= 2s + r \\
t &= s \\
z &= 3s + r.
\end{aligned}
\]
Solving for \(r, s\), we find our solution is given by
\[
u(x, t) = z(r(x, t), s(x, t)) = x + t.
\]

2. Solve
\[
\begin{aligned}
&u_t + u_x^2 + u = 0 \\
&u(x, 0) = x.
\end{aligned}
\]
Answer: Let
\[
F = q + p^2 + z.
\]
The set of characteristic equations is given by

\[
\begin{align*}
\frac{dx}{ds} &= 2p \\
x(r, 0) &= r \\
\frac{dt}{ds} &= 1 \\
t(r, 0) &= 0 \\
\frac{dz}{ds} &= q + 2p^2 \\
z(r, 0) &= r \\
\frac{dp}{ds} &= -p \\
p(r, 0) &= \psi_1(r) \\
\frac{dq}{ds} &= -q \\
z(r, 0) &= \psi_2(r)
\end{align*}
\]

where \(\psi_1\) and \(\psi_2\) satisfy

\[
\begin{align*}
\phi' &= \psi_1 \gamma_1' + \psi_2 \gamma_2' \\
\psi_2 + \psi_1^2 + \phi &= 0.
\end{align*}
\]

Therefore, we conclude that \(\psi_1 = 1\) and \(\psi_2 = -1 - r\). Solving our system of equations, we get

\[
\begin{align*}
p &= e^{-s} \\
q &= (-1 - r)e^{-s} \\
x &= -2e^{-s} + 2 + r \\
t &= s \\
z &= -e^{-2s} + (1 + r)e^{-s}.
\end{align*}
\]

Solving for \(r\) and \(s\), we see that \(s = t\), \(r = x + 2e^{-t} - 2\). Therefore, we conclude that our solution is given by

\[
u(x, t) = -e^{-2t} + (1 + x + 2e^{-t} - 2)e^{-t}
\]

or

\[
u(x, t) = (x + e^{-t} - 1)e^{-t}.
\]

3. Assume \((\vec{x}(\vec{r}, s), z(\vec{r}, s), \vec{p}(\vec{r}, s))\) is the solution of the characteristic ODEs for the fully nonlinear first-order equation

\[
\begin{align*}
F(\vec{x}, u, Du) &= 0 \\
u|\Gamma &= \phi
\end{align*}
\]

which satisfies the initial condition \((\vec{x}(\vec{r}, 0), z(\vec{r}, 0), \vec{p}(\vec{r}, 0)) = (\Gamma(\vec{r}), \phi(\vec{r}), \Psi(\vec{r})),\) where \((\Gamma, \phi, \Psi)\) is admissible initial data. Show that

\[
\frac{d}{ds} F(\vec{x}, z, \vec{p}) = 0.
\]

Note: This result proves part of the local existence theorem.

Answer: Let

\[
f(s) = F(\vec{x}(\vec{r}, s), z(\vec{r}, s), \vec{p}(\vec{r}, s)).
\]
By the chain rule,
\[ \frac{df}{ds} = \sum_{i=1}^{n} F_{x_i} \frac{\partial x_i}{\partial s} + F_z \frac{\partial z}{\partial s} + \sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial s}. \]

Then using the characteristic equations that \( \vec{x}, z, \vec{p} \) satisfy, we conclude that
\[ \frac{df}{ds} = \sum_{i=1}^{n} F_{x_i} F_{p_i} + F_z \sum_{i=1}^{n} p_i F_{p_i} + \sum_{i=1}^{n} F_{p_i} [-F_{x_i} - p_i F_z] = 0. \]

**Remark:** Then using the assumption that \( f(0) = 0 \), we conclude that \( f(s) = 0 \).

4. Consider the initial-value problem
\[
\begin{aligned}
\left\{ \begin{array}{ll}
  u_t + au_x = 0 & -\infty < x < \infty, t > 0 \\
  u(x, 0) = \phi(x)
\end{array} \right.
\end{aligned}
\]

We say \( u \) is a weak solution of (*) if \( u \) satisfies
\[
\int_0^\infty \int_{-\infty}^\infty u[v_t + av_x] \, dx \, dt + \int_{-\infty}^\infty \phi(x) v(x) \, dx = 0
\]
for all \( v \in C^\infty(\mathbb{R}^n \times [0, \infty)) \) with compact support. Assume that \( \phi \) is a piecewise \( C^1 \) function. Show that \( u(x, t) = \phi(x - at) \) is a weak solution of (*).

**Answer:** We assume that \( \phi \) just has one jump discontinuity. We can use a similar argument if \( \phi \) has an arbitrary number of discontinuities. Suppose \( \phi \) has a jump discontinuity at \( x_0 \). Let \( u(x, t) = u^-(x, t) = \phi(x - at) \) to the left of the curve \( x - at = x_0 \) and let \( u(x, t) = u^+(x, t) = \phi(x - at) \) to the right of the curve \( x - at = x_0 \). Let \( \Omega^- \) be the region to the left of the curve of discontinuity and \( \Omega^+ \) be the region to the right. Under the assumption that \( \phi \) has compact support, then we know \( u(x, t) = \phi(x - at) \) will have compact support. By our integration-by-parts formula, we know that
\[
\int_{\Omega^-} u[v_t + av_x] \, dx \, dt = -\int_{\Omega^-} [u_t + au_x]v \, dx \, dt + \int_{\partial \Omega^-} [uv \nu_2 + auv \nu_1] \, ds
\]
where \( \vec{\nu} = (\nu_1, \nu_2) \) is the outward unit normal to \( \Omega^- \). On \( x - at = x_0 \), we calculate that \( \vec{\nu} = (1 + a^2)^{-1/2}(1, -a) \). On \( t = 0 \), we calculate that \( \vec{\nu} = (0, -1) \). Therefore, we conclude that
\[
\int_{\partial \Omega^-} [uv \nu_2 + auv \nu_1] \, ds = \int_{x - at = x_0} (1 + a^2)^{-1/2}[-auv + avu] \, ds + \int_{t=0} [uv] \, ds
\]
\[
= \int_{-\infty}^{x_0} u(x, 0) v(x, 0) \, dx = -\int_{-\infty}^{x_0} \phi(x) v(x, 0) \, dx.
\]

Therefore, we conclude that
\[
\int_{\Omega^-} u[v_t + av_x] \, dx \, dt = -\int_{\Omega^-} [u_t + au_x]v \, dx \, dt - \int_{-\infty}^{x_0} \phi(x) v(x, 0) \, dx.
\]
Then using the fact that \( u(x,t) = \phi(x-\alpha t) \) is smooth in \( \Omega^- \), we can conclude that \( u_t + \alpha u_x = -a\phi'(x-\alpha t) + a\phi'(x-\alpha t) = 0 \) for \( (x,t) \in \Omega^- \). Therefore, we conclude that
\[
\iint_{\Omega^-} u[v_t + av_x] \, dx \, dt = -\int_{-\infty}^{x_0} \phi(x)v(x,0) \, dx.
\]
Similarly,
\[
\iint_{\Omega^+} u[v_t + av_x] \, dx \, dt = -\int_{x_0}^{\infty} \phi(x)v(x,0) \, dx.
\]
Therefore, we conclude that
\[
\int_0^t \int_{-\infty}^{\infty} u[v_t + av_x] \, dx \, dt + \int_{x_0}^{\infty} \phi(x)v(x,0) \, dx = 0,
\]
meaning \( u \) is a weak solution.

5. Consider the initial-value problem
\[
(\star) \begin{cases} [g(u)]_t + [f(u)]_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \end{cases}
\]
We say \( u \) is a weak solution of (\star) if \( u \) satisfies
\[
\int_0^\infty \int_{-\infty}^{\infty} g(u)v_t + f(u)v_x \, dx \, dt + \int_{-\infty}^{\infty} g(\phi(x))v(x,0) \, dx = 0
\]
for all \( v \in C^\infty(\mathbb{R} \times [0,\infty)) \) with compact support. Suppose \( u \) is a weak solution of (\star) such that \( u \) has a jump discontinuity across the curve \( x = \xi(t) \), but \( u \) is smooth on either side of the curve \( x = \xi(t) \). Let \( u^-(x,t) \) be the value of \( u \) to the left of the curve and \( u^+(x,t) \) be the value of \( u \) to the right of the curve. Prove that \( u \) must satisfy the condition
\[
\frac{[f(u)]}{[g(u)]} = \xi'(t)
\]
across the curve of discontinuity, where
\[
[f(u)] = f(u^-) - f(u^+), \quad [g(u)] = g(u^-) - g(u^+).
\]

**Answer:** If \( u \) is a weak solution of (\star), then
\[
\int_0^\infty \int_{-\infty}^{\infty} [g(u)v_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^{\infty} g(\phi(x))v(x,0) \, dx = 0
\]
for all smooth functions \( v \in C^\infty(\mathbb{R} \times [0,\infty)) \) with compact support. Let \( v \) be a smooth function such that \( v(x,0) = 0 \), and break up the first integral into the regions \( \Omega^- \), \( \Omega^+ \) where
\[
\Omega^- \equiv \{(x,t) : 0 < t < \infty, -\infty < x < \xi(t)\} \\
\Omega^+ \equiv \{(x,t) : 0 < t < \infty, \xi(t) < x < +\infty\}.
\]
Therefore,
\[ 0 = \int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x] \, dx \, dt + \int_{-\infty}^\infty g(\phi(x))v(x,0) \, dx \]
\[ = \iint_{\Omega^-} [g(u)v_t + f(u)v_x] \, dx \, dt + \iint_{\Omega^+} [g(u)v_t + f(u)v_x] \, dx \, dt. \]

Combining the Divergence Theorem with the fact that \( v \) has compact support and \( v(x,0) = 0 \), we have
\[ \iint_{\Omega^-} [g(u)v_t + f(u)v_x] \, dx \, dt = -\iint_{\Omega^-} [(g(u))_t + (f(u))_x]v \, dx \, dt \]
\[ + \int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1] \, ds \]
where \( \nu = (\nu_1, \nu_2) \) is the outward unit normal to \( \Omega^- \).

Similarly, we see that
\[ \iint_{\Omega^+} [g(u)v_t + f(u)v_x] \, dx \, dt = -\iint_{\Omega^+} [(g(u))_t + (f(u))_x]v \, dx \, dt \]
\[ - \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1] \, ds. \]

By assumption, \( u \) is a weak solution of
\[ [g(u)]_t + [f(u)]_x = 0 \]
and \( u \) is smooth on either side of \( x = \xi(t) \). Therefore, \( u \) is a strong solution on either side of the curve of discontinuity. Consequently, we see that
\[ \iint_{\Omega^-} [(g(u))_t + (f(u))_x]v \, dx \, dt = 0 = \iint_{\Omega^+} [(g(u))_t + (f(u))_x]v \, dx \, dt. \]

Combining these facts, we see that
\[ \int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1] \, ds - \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1] \, ds = 0. \]
Since this is true for all smooth functions \( v \), we have
\[
g(u^-)\nu_2 + f(u^-)\nu_1 = g(u^+)\nu_2 + f(u^+)\nu_1,
\]
which implies
\[
\frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = -\frac{\nu_2}{\nu_1}.
\]
Now the curve \( x = \xi(t) \) has slope given by the negative reciprocal of the normal to the curve; that is,
\[
\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2}.
\]
Therefore,
\[
\xi'(t) = -\frac{\nu_2}{\nu_1} = \frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = \frac{[f(u)]}{[g(u)]},
\]
as claimed.