Math 220A - Fall 2002
Homework 2 Solutions

1. Solve
$$\begin{cases} u_x^2 u_t - 1 = 0 \\ u(x,0) = x. \end{cases}$$

Answer: Let
$$F(p,q,z,x,t) = p^2 q - 1.$$
The set of characteristic equations are given by
$$\begin{array}{ll} \frac{dx}{ds} = 2pq & x(r,0) = r \\ \frac{dt}{ds} = p^2 & t(r,0) = 0 \\ \frac{dz}{ds} = 3 & z(r,0) = r \\ \frac{dp}{ds} = 0 & p(r,0) = \psi_1(r) \\ \frac{dq}{ds} = 0 & q(r,0) = \psi_2(r) \end{array}$$
where ψ_1, ψ_2 satisfy
$$\begin{aligned} \phi'(r) &= \psi_1(r) \\ \psi_1^2 \psi_2 - 1 &= 0. \end{aligned}$$
Therefore,
$$\psi_1(r) = 1 = \psi_2(r).$$
Solving this system of ODEs, we have
$$\begin{aligned} p &= 1 \\ q &= 1 \\ x &= 2s + r \\ t &= s \\ z &= 3s + r. \end{aligned}$$
Solving for r, s, we find our solution is given by
$$\boxed{u(x,t) = z(r(x,t), s(x,t)) = x + t.}$$

2. Solve
$$\begin{cases} u_t + u_x^2 + u = 0 \\ u(x,0) = x. \end{cases}$$

Answer: Let
$$F = q + p^2 + z.$$

The set of characteristic equations is given by

$$\begin{aligned} \tfrac{dx}{ds} &= 2p & x(r,0) &= r \\ \tfrac{dt}{ds} &= 1 & t(r,0) &= 0 \\ \tfrac{dz}{ds} &= q + 2p^2 & z(r,0) &= r \\ \tfrac{dp}{ds} &= -p & p(r,0) &= \psi_1(r) \\ \tfrac{dq}{ds} &= -q & q(r,0) &= \psi_2(r) \end{aligned}$$

where ψ_1 and ψ_2 satisfy

$$\begin{aligned} \phi' &= \psi_1\gamma_1' + \psi_2\gamma_2' \\ \psi_2 &+ \psi_1^2 + \phi = 0. \end{aligned}$$

Therefore, we conclude that $\psi_1 = 1$ and $\psi_2 = -1-r$. Solving our system of equations, we get

$$\begin{aligned} p &= e^{-s} \\ q &= (-1-r)e^{-s} \\ x &= -2e^{-s} + 2 + r \\ t &= s \\ z &= -e^{-2s} + (1+r)e^{-s}. \end{aligned}$$

Solving for r and s, we see that $s = t$, $r = x + 2e^{-t} - 2$. Therefore, we conclude that our solution is given by

$$u(x,t) = -e^{-2t} + (1 + x + 2e^{-t} - 2)e^{-t}$$

or

$$\boxed{u(x,t) = (x + e^{-t} - 1)e^{-t}.}$$

3. Assume $(\vec{x}(\vec{r},s), z(\vec{r},s), \vec{p}(\vec{r},s))$ is the solution of the characteristic ODEs for the fully nonlinear first-order equation

$$\begin{cases} F(\vec{x}, u, Du) = 0 \\ u|_\Gamma = \phi \end{cases}$$

which satisfies the initial condition $(\vec{x}(\vec{r},0), z(\vec{r},0), \vec{p}(\vec{r},0)) = (\Gamma(\vec{r}), \phi(\vec{r}), \Psi(\vec{r}))$, where (Γ, ϕ, Ψ) is admissible initial data. Show that

$$\frac{d}{ds}F(\vec{x}, z, \vec{p}) = 0.$$

Note: This result proves part of the local existence theorem.

Answer: Let

$$f(s) = F(\vec{x}(\vec{r},s), z(\vec{r},s), \vec{p}(\vec{r},s)).$$

By the chain rule,

$$\frac{df}{ds} = \sum_{i=1}^{n} F_{x_i} \frac{\partial x_i}{\partial s} + F_z \frac{\partial z}{\partial s} + \sum_{i=1}^{n} F_{p_i} \frac{\partial p_i}{\partial s}.$$

Then using the characteristic equations that $\vec{x}, z, \vec{p}$ satisfy, we conclude that

$$\frac{df}{ds} = \sum_{i=1}^{n} F_{x_i} F_{p_i} + F_z \sum_{i=1}^{n} p_i F_{p_i} + \sum_{i=1}^{n} F_{p_i}[-F_{x_i} - p_i F_z] = 0.$$

Remark: Then using the assumption that $f(0) = 0$, we conclude that $f(s) = 0$.

4. Consider the initial-value problem

$$(*) \begin{cases} u_t + a u_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

We say u is a weak solution of (*) if u satisfies

$$\int_0^\infty \int_{-\infty}^\infty u[v_t + a v_x] \, dx \, dt + \int_{-\infty}^\infty \phi(x) v(x) \, dx = 0$$

for all $v \in C^\infty(\mathbb{R}^n \times [0,\infty))$ with compact support. Assume that ϕ is a piecewise C^1 function. Show that $u(x,t) = \phi(x - at)$ is a weak solution of (*).

Answer: We assume that ϕ just has one jump discontinuity. We can use a similar argument if ϕ has an arbitrary number of discontinuities. Suppose ϕ has a jump discontinuity at x_0. Let $u(x,t) = u^-(x,t) = \phi(x-at)$ to the left of the curve $x-at = x_0$ and let $u(x,t) = u^+(x,t) = \phi(x-at)$ to the right of the curve $x - at = x_0$. Let Ω^- be the region to the left of the curve of discontinuity and Ω^+ be the region to the right. Under the assumption that ϕ has compact support, then we know $u(x,t) = \phi(x - at)$ will have compact support. By our integration-by-parts formula, we know that

$$\iint_{\Omega^-} u[v_t + a v_x] \, dx \, dt = -\iint_{\Omega^-} [u_t + a u_x] v \, dx \, dt + \int_{\partial\Omega^-} [u v \nu_2 + a u v \nu_1] \, ds$$

where $\vec{\nu} = (\nu_1, \nu_2)$ is the outward unit normal to Ω^-. On $x - at = x_0$, we calculate that $\vec{\nu} = (1 + a^2)^{-1/2}(1, -a)$. On $t = 0$, we calculate that $\vec{\nu} = (0, -1)$. Therefore, we conclude that

$$\begin{aligned} \int_{\partial\Omega^-} [u v \nu_2 + a u v \nu_1] \, ds &= \int_{x-at=x_0} (1+a^2)^{-1/2}[-auv + auv] \, ds + \int_{t=0} [-uv] \, ds \\ &= \int_{-\infty}^{x_0} -u(x,0)v(x,0) \, dx = -\int_{-\infty}^{x_0} \phi(x) v(x,0) \, dx. \end{aligned}$$

Therefore, we conclude that

$$\iint_{\Omega^-} u[v_t + a v_x] \, dx \, dt = -\iint_{\Omega^-} [u_t + a u_x] v \, dx \, dt - \int_{-\infty}^{x_0} \phi(x) v(x,0) \, dx.$$

Then using the fact that $u(x,t) = \phi(x - at)$ is smooth in Ω^-, we can conclude that $u_t + au_x = -a\phi'(x - at) + a\phi'(x - at) = 0$ for $(x,t) \in \Omega^-$. Therefore, we conclude that

$$\iint_{\Omega^-} u[v_t + av_x]\, dx\, dt = -\int_{-\infty}^{x_0} \phi(x)v(x,0)\, dx.$$

Similarly,

$$\iint_{\Omega^+} u[v_t + av_x]\, dx\, dt = -\int_{x_0}^{\infty} \phi(x)v(x,0)\, dx.$$

Therefore, we conclude that

$$\int_0^t \int_{-\infty}^{\infty} u[v_t + av_x]\, dx\, dt + \int_0^{\infty} \phi(x)v(x,0)\, dx = 0,$$

meaning u is a weak solution.

5. Consider the initial-value problem

$$(*) \begin{cases} [g(u)]_t + [f(u)]_x = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \end{cases}$$

We say u is a weak solution of (*) if u satisfies

$$\int_0^{\infty} \int_{-\infty}^{\infty} g(u)v_t + f(u)v_x\, dx\, dt + \int_{-\infty}^{\infty} g(\phi(x))v(x,0)\, dx = 0$$

for all $v \in C^{\infty}(\mathbb{R} \times [0,\infty))$ with compact support. Suppose u is a weak solution of (*) such that u has a jump discontinuity across the curve $x = \xi(t)$, but u is smooth on either side of the curve $x = \xi(t)$. Let $u^-(x,t)$ be the value of u to the left of the curve and $u^+(x,t)$ be the value of u to the right of the curve. Prove that u must satisfy the condition

$$\frac{[f(u)]}{[g(u)]} = \xi'(t)$$

across the curve of discontinuity, where

$$[f(u)] = f(u^-) - f(u^+)$$
$$[g(u)] = g(u^-) - g(u^+).$$

Answer: If u is a weak solution of (*), then

$$\int_0^{\infty} \int_{-\infty}^{\infty} [g(u)v_t + f(u)v_x]\, dx\, dt + \int_{-\infty}^{\infty} g(\phi(x))v(x,0)\, dx = 0$$

for all smooth functions $v \in C^{\infty}(\mathbb{R} \times [0,\infty))$ with compact support. Let v be a smooth function such that $v(x,0) = 0$, and break up the first integral into the regions Ω^-, Ω^+ where

$$\Omega^- \equiv \{(x,t) : 0 < t < \infty,\ -\infty < x < \xi(t)\}$$
$$\Omega^+ \equiv \{(x,t) : 0 < t < \infty,\ \xi(t) < x < +\infty\}.$$

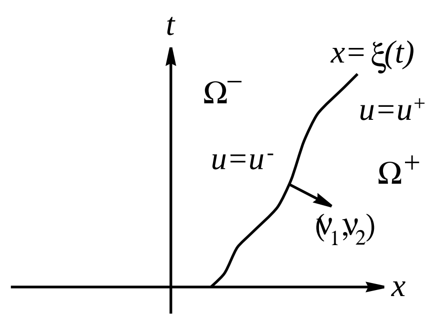
Therefore,

$$0 = \int_0^\infty \int_{-\infty}^\infty [g(u)v_t + f(u)v_x]\, dx\, dt + \int_{-\infty}^\infty g(\phi(x))v(x,0)\, dx$$
$$= \iint_{\Omega^-} [g(u)v_t + f(u)v_x]\, dx\, dt + \iint_{\Omega^+} [g(u)v_t + f(u)v_x]\, dx\, dt.$$

Combining the Divergence Theorem with the fact that v has compact support and $v(x,0) = 0$, we have

$$\iint_{\Omega^-} [g(u)v_t + f(u)v_x]\, dx\, dt = -\iint_{\Omega^-} [(g(u))_t + (f(u))_x]v\, dx\, dt$$
$$+ \int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1]\, ds$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to Ω^-.



Similarly, we see that

$$\iint_{\Omega^+} [g(u)v_t + f(u)v_x]\, dx\, dt = -\iint_{\Omega^+} [(g(u))_t + (f(u))_x]v\, dx\, dt$$
$$- \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1]\, ds.$$

By assumption, u is a weak solution of

$$[g(u)]_t + [f(u)]_x = 0$$

and u is smooth on either side of $x = \xi(t)$. Therefore, u is a strong solution on either side of the curve of discontinuity. Consequently, we see that

$$\iint_{\Omega^-} [(g(u))_t + (f(u))_x]v\, dx\, dt = 0 = \iint_{\Omega^+} [(g(u))_t + (f(u))_x]v\, dx\, dt.$$

Combining these facts, we see that

$$\int_{x=\xi(t)} [g(u^-)v\nu_2 + f(u^-)v\nu_1]\, ds - \int_{x=\xi(t)} [g(u^+)v\nu_2 + f(u^+)v\nu_1]\, ds = 0.$$

Since this is true for all smooth functions v, we have

$$g(u^-)\nu_2 + f(u^-)\nu_1 = g(u^+)\nu_2 + f(u^+)\nu_1,$$

which implies

$$\frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = -\frac{\nu_2}{\nu_1}.$$

Now the curve $x = \xi(t)$ has slope given by the negative reciprocal of the normal to the curve; that is,

$$\frac{dt}{dx} = \frac{1}{\xi'(t)} = -\frac{\nu_1}{\nu_2}.$$

Therefore,

$$\xi'(t) = -\frac{\nu_2}{\nu_1} = \frac{f(u^-) - f(u^+)}{g(u^-) - g(u^+)} = \frac{[f(u)]}{[g(u)]},$$

as claimed.