Math 220A - Fall 2002
Homework 3 Solutions

1. Find the unique weak solution of

$$\begin{cases} u_t + uu_x = 0, \ t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

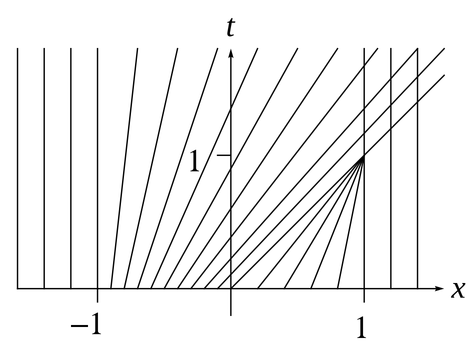
where

$$\phi(x) = \begin{cases} 0 & \text{for } x \leq -1 \\ x+1 & \text{for } -1 \leq x \leq 0 \\ -x+1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{for } x \geq 1, \end{cases}$$

which satisfies the Rankine-Hugoniot condition and the entropy condition. Show that your solution satisfies the entropy condition. Draw a picture describing your answer, showing the projected characteristics and any shock curves.

Answer: Using the method of characteristics, we see that the projected characteristics are given by the curves $x = \phi(r)t + r$, and, that u is constant along these curves. In particular, we see that the projected characteristic curves are given by

$$\begin{aligned} &r < -1 \implies \phi(r) = 0 \implies x = r \\ &-1 < r < 0 \implies \phi(r) = r+1 \implies x = (r+1)t + r \\ &0 < r < 1 \implies \phi(r) = -r+1 \implies x = (-r+1)t + r \\ &r > 1 \implies \phi(r) = 0 \implies x = r. \end{aligned}$$



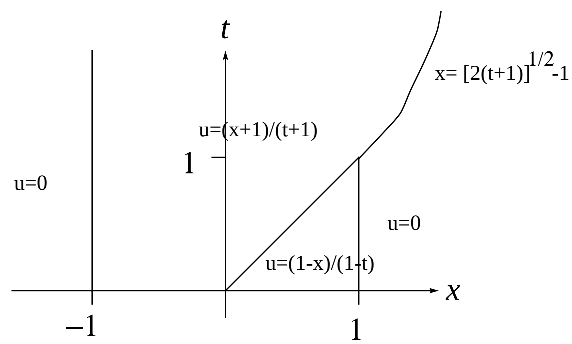
We notice that these curves do not intersect for $t < 1$. Therefore, for $\boxed{0 \leq t \leq 1}$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 0 & x < -1 \\ \dfrac{x+1}{t+1} & -1 < x < t \\ \dfrac{1-x}{1-t} & t < x < 1 \\ 0 & x > 1 \end{cases}}$$

Now at $t = 1$, the projected characteristics intersect. We will introduce a shock curve. Our solution should satisfy $u^- = (x+1)/(t+1)$ and $u^+ = 0$. Using the RH jump condition, we have

$$\begin{aligned}\xi'(t) &= \frac{\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2}}{u^- - u^+} \\ &= \frac{(x+1)}{2(t+1)}\end{aligned}$$

Solving this differential equation, we see that the equation for the curve of discontinuity is given by $x = \sqrt{2(t+1)} - 1$. Notice that $f'(u) = u$, and, $u^- = (x+1)/(t+1) > 0 = u^+$. Therefore, the entropy condition is satisfied across this curve of discontinuity.



Finally, we conclude that for $\boxed{t \geq 1}$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 0 & x < -1 \\ \dfrac{x+1}{t+1} & -1 < x < \sqrt{2(t+1)} - 1 \\ 0 & x > \sqrt{2(t+1)} - 1. \end{cases}}$$

2. Find the unique weak solution of

$$\begin{cases} \left(\dfrac{u^2}{2}\right)_t + \left(\dfrac{u^3}{3}\right)_x = 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} 1 & x < 0 \\ 0 & x > 0 \end{cases}$$

Answer: We recall that in order for a function u to be a weak solution of an equation of the form

$$[g(u)]_t + [f(u)]_x = 0,$$

we need u to satisfy the jump condition

$$\xi'(t) = \frac{[f(u)]}{[g(u)]}$$

across any curve of discontinuity. Here $f(u) = u^3/3$, $g(u) = u^2/2$. We want $u^- = 1$, $u^+ = 0$. Therefore, the curve of discontinuity must satisfy

$$\xi'(t) = \frac{\frac{1}{3}}{\frac{1}{2}} = \frac{2}{3}.$$

Therefore, we define u such that

$$\boxed{u(x,t) = \begin{cases} 1 & x < \frac{2}{3}t \\ 0 & x > \frac{2}{3}t. \end{cases}}$$

3. Find the unique weak solution of

$$\begin{cases} u_t + \left(\frac{u^2}{2}\right)_x = 0 \ , t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

which satisfies the Rankine-Hugoniot jump condition and the entropy condition, where the initial data

$$\phi(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 3 & \text{if } x > 0. \end{cases}$$

Answer: Using the method of characteristics, we see that the projected characteristics satisfy $x = \phi(r)t + r$. In particular, we have

$$r < -1 \implies \phi(r) = 1 \implies x = t + r$$
$$-1 < r < 0 \implies \phi(r) = 0 \implies x = r$$
$$r > 0 \implies \phi(r) = 3 \implies x = 3t + r.$$

We notice that projected characteristics intersect immediately at $x = -1$. Therefore, we need to put in a shock curve. We want $u^- = 1$, $u^+ = 0$. Therefore, using the RH jump condition, we see that the shock curve $\xi_1(t)$ must satisfy

$$\begin{aligned} \xi_1'(t) &= \frac{\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2}}{u^- - u^+} \\ &= \frac{1}{2}. \end{aligned}$$

Therefore, $\xi_1(t) = \frac{1}{2}t - 1$. We note that the entropy condition is satisfied along this curve of discontinuity, because

$$\boxed{f'(u^-) = u^- = 1 > \frac{1}{2} = \sigma > 0 = u^+ = f'(u^+).}$$

Next, we need to put a rarefaction wave in between $x = 0$ and $x = 3t$. This rarefaction wave is given by $u(x,t) = G(x/t) = (f')^{-1}(x/t) = x/t$. Therefore, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 1 & x < \dfrac{1}{2}t - 1 \\ 0 & \dfrac{1}{2}t - 1 < x < 0 \\ \dfrac{x}{t} & 0 < x < 3t \\ 3 & x > 3t \end{cases}}$$

until the shock curve hits the rarefaction wave at $t = 2$. Therefore, the solution above is valid for $\boxed{0 \leq t \leq 2}$.

After this time t, we need to put in a new shock curve which will satisfy the RH jump condition. We want $u^- = 1$ and $u^+ = x/t$. Therefore, by the RH jump condition, we have

$$\begin{aligned} \xi_2'(t) &= \frac{\frac{(u^-)^2}{2} - \frac{(u^+)^2}{2}}{u^- - u^+} \\ &= \frac{\frac{1}{2} - \frac{x^2}{2t^2}}{1 - \frac{x}{t}}. \end{aligned}$$

Simplifying, we arrive at the differential equation

$$\frac{dx}{dt} = \frac{1}{2}\left(1 + \frac{x}{t}\right).$$

This is a linear ODE which can be rewritten as

$$\left(\frac{1}{\sqrt{t}}x\right)' = \frac{1}{2\sqrt{t}},$$

using the integrating factor $t^{-1/2}$. Solving this ODE and using the initial condition $x = 0$, $t = 2$, we have

$$\xi_2(t) = t - \sqrt{2t}.$$

Therefore, for $\boxed{t \geq 2}$, our solution is given by

$$\boxed{u(x,t) = \begin{cases} 1 & x < t - \sqrt{2t} \\ \dfrac{x}{t} & t - \sqrt{2t} < x < 3t \\ 3 & x > 3t \end{cases}}$$

4. Consider the following initial-value problem

$$\begin{cases} u_t - (\cos u)_x = 0 \\ u(x,0) = \phi(x). \end{cases}$$

Find the unique, weak admissible solution which satisfies the Oleinik entropy condition if the initial conditions are given by

(a)

$$\phi(x) = \begin{cases} \frac{\pi}{2} & x < 0 \\ -\frac{\pi}{2} & x > 0 \end{cases}$$

Answer: Using the method of characteristics, we see that the projected characteristics are given by $x = \sin(\phi(r))t + r$. Therefore, we have

$$\begin{aligned} r < 0 &\implies x = t + r \\ r > 0 &\implies x = -t + r. \end{aligned}$$

In particular, the projected characteristics intersect for $t > 0$. By drawing a graph of the function $f(u) = -\cos(u)$, we see that $f(u) = -\cos(u), u^- = \pi/2, u^+ = -\pi/2$ satisfy the Oleinik entropy condition,

$$\frac{f(u^-) - f(u^+)}{u^- - u^+} \leq \frac{f(u^-) - f(u)}{u^- - u} \qquad \forall u \in (u^+, u^-).$$

Therefore, a shock curve is admissible. We know the shock curve must satisfy the RH jump condition. Therefore, we need

$$\xi'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{-\cos(\pi/2) + \cos(-\pi/2)}{\pi/2 + \pi/2} = 0.$$

Therefore, we conclude that the unique, weak, admissible solution is given by

$$\boxed{u(x,t) = \begin{cases} \frac{\pi}{2} & x < 0 \\ -\frac{\pi}{2} & x > 0. \end{cases}}$$

(b)

$$\phi(x) = \begin{cases} \pi & x < 0 \\ -\frac{\pi}{2} & x > 0 \end{cases}$$

Answer: As in part (a), we know the projected characteristics are given by $x = \sin(\phi(r))t + r$. Therefore,

$$\begin{aligned} r < 0 &\implies x = r \\ r > 0 &\implies x = -t + r. \end{aligned}$$

Therefore, the projected characteristics intersect for $t > 0$. However, upon looking at the graph of the function $f(u) = -\cos(u)$ between $u^+ = -\pi/2$ and $u^- = \pi$, we see that the Oleinik entropy condition is not satisfied. Therefore, we cannot prove in a shock between u^- and u^+. We must use the rubberband method.

In particular, choose u_2 such that

$$f'(u_2) = \frac{f(u_2) - f(u^+)}{u_2 - u^+},$$

or, rewritten as

$$\boxed{\sin(u_2) = \frac{-\cos(u_2)}{u_2 + \frac{\pi}{2}}.}$$

Then a shock curve will be admissible from u_2 to u^+ as the Oleinik entropy condition will be satisfied in this region. This curve will be determined by the RH jump condition. In particular, we will have

$$\begin{aligned}\xi'(t) &= \frac{f(u_2) - f(u^+)}{u_2 - u^+} \\ &= \frac{-\cos(u_2)}{u_2 + \frac{\pi}{2}}.\end{aligned}$$

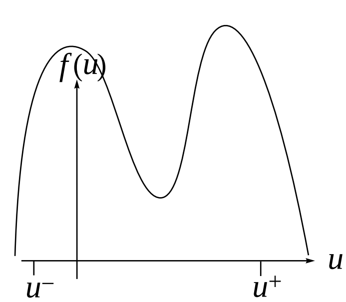
Further, we notice that $f(u) = -\cos(u)$ will be invertible in the region $(u_2, u^-) = (u_2, \pi)$. Therefore, we can put in a rarefaction wave to go from π to u_2.

Therefore, our solution will be defined as follows,

$$\boxed{u(x,t) = \begin{cases} \pi & x < 0 \\ G\left(\frac{x}{t}\right) & 0 < x < \sin(u_2)t \\ -\frac{\pi}{2} & x > \sin(u_2)t \end{cases}}$$

where $G(x/t) = (f')^{-1}(x/t) = \sin^{-1}(x/t)$ where $\sin(y)$ is restricted to the region u_2 to π.

5. Consider f, u^-, u^+ shown below.



Consider the initial-value problem

$$\begin{cases} u_t + [f(u)]_x = 0, & t \geq 0 \\ u(x,0) = \phi(x) \end{cases}$$

where

$$\phi(x) = \begin{cases} u^- & x < 0 \\ u^+ & x > 0. \end{cases}$$

Find the weak solution which satisfies the Oleinik entropy condition.

Answer: We will use the rubberband method. (See picture below.)

Consider u_2 such that

$$f'(u_2) = \frac{f(u^-) - f(u_2)}{u^- - u_2}.$$

From the graph we see that for all $u \in (u^-, u_2)$,

$$\frac{f(u^-) - f(u_2)}{u^- - u_2} \leq \frac{f(u^-) - f(u)}{u^- - u}.$$

Therefore, the Oleinik entropy condition is satisfied for f, u^- and u_2. In addition, if we put in a curve of discontinuity along the curve $x = f'(u_2)t$ such that u jumps from u^- to u_2, we see that the RH jump condition is satisfied because $dx/dt = f'(u_2) = [f(u)]/[u]$.

Next, consider u_3 such that

$$f'(u_3) = \frac{f(u_3) - f(u^+)}{u_3 - u^+}.$$

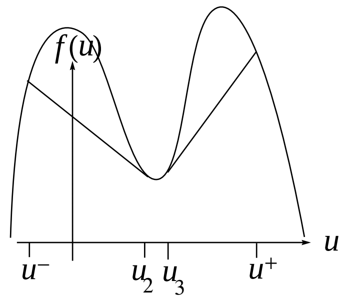
We see that for $u \in (u_3, u^+)$,

$$\frac{f(u_3) - f(u^+)}{u_3 - u^+} \leq \frac{f(u_3) - f(u)}{u_3 - u}.$$

Therefore, the Oleinik entropy condition is satisfied for f, u_3 and u^+. In addition, the RH jump condition is satisfied if we define $x = f'(u_3)t$ as the curve of discontinuity such that u jumps from u_3 to u^+ along this curve, because $dx/dt = f'(u_3) = [f(u)]/[u]$. In addition, f' is strictly increasing in the interval (u_3, u_2). Therefore, f' is invertible on that interval. Therefore, we can put a rarefaction wave between u_3 and u_2. Therefore, our solution is defined as

$$\boxed{u(x,t) = \begin{cases} u^- & x < f'(u_2)t \\ G\left(\frac{x}{t}\right) & f'(u_2)t < x < f'(u_3)t \\ u^+ & x > f'(u_3)t \end{cases}}$$

where $G = (f')^{-1}$ for f restricted to the interval (u_2, u_3).


$f(u)$
u
u^-
u_2
u_3
u^+