1. Find the unique weak solution of

\[
\begin{cases}
  u_t + uu_x = 0, \quad t \geq 0 \\
  u(x,0) = \phi(x)
\end{cases}
\]

where

\[
\phi(x) = \begin{cases}
  0 & \text{for } x \leq -1 \\
  x + 1 & \text{for } -1 \leq x \leq 0 \\
  -x + 1 & \text{for } 0 \leq x \leq 1 \\
  0 & \text{for } x \geq 1,
\end{cases}
\]

which satisfies the Rankine-Hugoniot condition and the entropy condition. Show that your solution satisfies the entropy condition. Draw a picture describing your answer, showing the projected characteristics and any shock curves.

**Answer:** Using the method of characteristics, we see that the projected characteristics are given by the curves \( x = \phi(r)t + r \), and, that \( u \) is constant along these curves. In particular, we see that the projected characteristic curves are given by

- \( r < -1 \implies \phi(r) = 0 \implies x = r \)
- \( -1 < r < 0 \implies \phi(r) = r + 1 \implies x = (r + 1)t + r \)
- \( 0 < r < 1 \implies \phi(r) = -r + 1 \implies x = (-r + 1)t + r \)
- \( r > 1 \implies \phi(r) = 0 \implies x = r \).

We notice that these curves do not intersect for \( t < 1 \). Therefore, for \( 0 \leq t \leq 1 \), our solution is given by

\[
u(x,t) = \begin{cases}
  0 & x < -1 \\
  \frac{x + 1}{t + 1} & -1 < x < t \\
  1 - x & t < x < 1 \\
  \frac{1 - x}{1 - t} & t < x < 1 \\
  0 & x > 1
\end{cases}
\]
Now at $t = 1$, the projected characteristics intersect. We will introduce a shock curve. Our solution should satisfy $u^- = (x + 1)/(t + 1)$ and $u^+ = 0$. Using the RH jump condition, we have

$$\dot{\xi}(t) = \frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \frac{(x + 1)}{2(t + 1)}$$

Solving this differential equation, we see that the equation for the curve of discontinuity is given by $x = \sqrt{2(t + 1)} - 1$. Notice that $f'(u) = u$, and, $u^- = (x + 1)/(t + 1) > 0 = u^+$. Therefore, the entropy condition is satisfied across this curve of discontinuity.

Finally, we conclude that for $t \geq 1$, our solution is given by

$$u(x, t) = \begin{cases} 
0 & x < -1 \\
\frac{x + 1}{t + 1} & -1 < x < \sqrt{2(t + 1)} - 1 \\
0 & x > \sqrt{2(t + 1)} - 1.
\end{cases}$$

2. Find the unique weak solution of

$$\begin{cases} 
\left(\frac{u^2}{2}\right)_t + \left(\frac{u^3}{3}\right)_x = 0 \\
u(x, 0) = \phi(x)
\end{cases}$$

where

$$\phi(x) = \begin{cases} 
1 & x < 0 \\
0 & x > 0
\end{cases}$$

**Answer:** We recall that in order for a function $u$ to be a weak solution of an equation of the form

$$[g(u)]_t + [f(u)]_x = 0,$$
we need $u$ to satisfy the jump condition

$$
\xi'(t) = \frac{[f(u)]}{[g(u)]}
$$

across any curve of discontinuity. Here $f(u) = u^3/3$, $g(u) = u^2/2$. We want $u^- = 1$, $u^+ = 0$. Therefore, the curve of discontinuity must satisfy

$$
\xi'(t) = \frac{1/3}{1/2} = \frac{2}{3}.
$$

Therefore, we define $u$ such that

$$
u(x,t) = \begin{cases} 
1 & x < \frac{2}{3}t \\
0 & x > \frac{2}{3}t.
\end{cases}
$$

3. Find the unique weak solution of

$$
\begin{cases}
u_t + \left(\frac{\nu^2}{2}\right)_x = 0, \quad t \geq 0 \\
u(x,0) = \phi(x)
\end{cases}
$$

which satisfies the Rankine-Hugoniot jump condition and the entropy condition, where the initial data

$$
\phi(x) = \begin{cases} 
1 & \text{if } x < -1 \\
0 & \text{if } -1 < x < 0 \\
3 & \text{if } x > 0.
\end{cases}
$$

**Answer:** Using the method of characteristics, we see that the projected characteristics satisfy $x = \phi(r)t + r$. In particular, we have

$$
\begin{align*}
    r < -1 & \implies \phi(r) = 1 \implies x = t + r \\
    -1 < r < 0 & \implies \phi(r) = 0 \implies x = r \\
    r > 0 & \implies \phi(r) = 3 \implies x = 3t + r.
\end{align*}
$$

We notice that projected characteristics intersect immediately at $x = -1$. Therefore, we need to put in a shock curve. We want $u^- = 1$, $u^+ = 0$. Therefore, using the RH jump condition, we see that the shock curve $\xi_1(t)$ must satisfy

$$
\xi_1'(t) = \frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} \quad \frac{1}{u^- - u^+} = \frac{1}{2}.
$$
Therefore, \( \xi_1(t) = \frac{1}{2}t - 1 \). We note that the entropy condition is satisfied along this curve of discontinuity, because

\[
\frac{d}{dt}(u^-) = u^- - \sigma > 0 = u^+ = \frac{d}{dt}(u^+).
\]

Next, we need to put a rarefaction wave in between \( x = 0 \) and \( x = 3t \). This rarefaction wave is given by \( u(x, t) = G(x/t) = (f')^{-1}(x/t) = x/t \). Therefore, our solution is given by

\[
u(x, t) = \begin{cases} 
1 & x < \frac{1}{2}t - 1 \\
0 & \frac{1}{2}t - 1 < x < 0 \\
x/t & 0 < x < 3t \\
3 & x > 3t
\end{cases}
\]

until the shock curve hits the rarefaction wave at \( t = 2 \). Therefore, the solution above is valid for \( 0 \leq t \leq 2 \).

After this time \( t \), we need to put in a new shock curve which will satisfy the RH jump condition. We want \( u^- = 1 \) and \( u^+ = x/t \). Therefore, by the RH jump condition, we have

\[
\xi_2'(t) = \frac{(u^-)^2}{2} - \frac{(u^+)^2}{2} = \frac{1}{2} - \frac{x^2}{2t^2}.
\]

Simplifying, we arrive at the differential equation

\[
\frac{dx}{dt} = \frac{1}{2} \left( 1 + \frac{x}{t} \right).
\]

This is a linear ODE which can be rewritten as

\[
\left( \frac{1}{\sqrt{t}} x \right)' = \frac{1}{2\sqrt{t}},
\]

using the integrating factor \( t^{-1/2} \). Solving this ODE and using the initial condition \( x = 0, t = 2 \), we have

\[
\xi_2(t) = t - \sqrt{2t}.
\]

Therefore, for \( t \geq 2 \), our solution is given by

\[
u(x, t) = \begin{cases} 
1 & x < t - \sqrt{2t} \\
x/t & t - \sqrt{2t} < x < 3t \\
3 & x > 3t
\end{cases}
\]
4. Consider the following initial-value problem
\[
\begin{align*}
&u_t - (\cos u)_x = 0 \\
&u(x,0) = \phi(x).
\end{align*}
\]
Find the unique, weak admissible solution which satisfies the Oleinik entropy condition if the initial conditions are given by

(a) \[
\phi(x) = \begin{cases} 
\frac{\pi}{2} & x < 0 \\
-\frac{\pi}{2} & x > 0 
\end{cases}
\]

Answer: Using the method of characteristics, we see that the projected characteristics are given by \( x = \sin(\phi(r)) t + r \). Therefore, we have
\[
\begin{align*}
& r < 0 \implies x = t + r \\
& r > 0 \implies x = -t + r.
\end{align*}
\]
In particular, the projected characteristics intersect for \( t > 0 \). By drawing a graph of the function \( f(u) = -\cos(u) \), we see that \( f(u) = -\cos(u), u^- = \pi/2, u^+ = -\pi/2 \) satisfy the Oleinik entropy condition,
\[
\frac{f(u^-) - f(u^+)}{u^- - u^+} \leq \frac{f(u^-) - f(u)}{u^- - u} \quad \forall u \in (u^+, u^-).
\]
Therefore, a shock curve is admissible. We know the shock curve must satisfy the RH jump condition. Therefore, we need
\[
\xi'(t) = \frac{f(u^-) - f(u^+)}{u^- - u^+} = \frac{-\cos(\pi/2) + \cos(-\pi/2)}{\pi/2 + \pi/2} = 0.
\]
Therefore, we conclude that the unique, weak, admissible solution is given by
\[
\begin{align*}
u(x, t) &= \begin{cases} 
\frac{\pi}{2} & x < 0 \\
-\frac{\pi}{2} & x > 0 
\end{cases}
\end{align*}
\]

(b) \[
\phi(x) = \begin{cases} 
\pi & x < 0 \\
-\pi & x > 0 
\end{cases}
\]

Answer: As in part (a), we know the projected characteristics are given by \( x = \sin(\phi(r)) t + r \). Therefore,
\[
\begin{align*}
r < 0 \implies x = r \\
r > 0 \implies x = -t + r.
\end{align*}
\]
Therefore, the projected characteristics intersect for \( t > 0 \). However, upon looking at the graph of the function \( f(u) = -\cos(u) \) between \( u^+ = -\pi/2 \) and \( u^- = \pi \), we see that the Oleinik entropy condition is not satisfied. Therefore, we cannot prove in a shock between \( u^- \) and \( u^+ \). We must use the rubberband method.

In particular, choose \( u_2 \) such that

\[
f'(u_2) = \frac{f(u_2) - f(u^+)}{u_2 - u^+},
\]

or, rewritten as

\[
\sin(u_2) = \frac{-\cos(u_2)}{u_2 + \frac{\pi}{2}}.
\]

Then a shock curve will be admissible from \( u_2 \) to \( u^+ \) as the Oleinik entropy condition will be satisfied in this region. This curve will be determined by the RH jump condition. In particular, we will have

\[
\xi'(t) = \frac{f(u_2) - f(u^+)}{u_2 - u^+} = \frac{-\cos(u_2)}{u_2 + \frac{\pi}{2}}.
\]

Further, we notice that \( f(u) = -\cos(u) \) will be invertible in the region \((u_2, u^-) = (u_2, \pi)\). Therefore, we can put in a rarefaction wave to go from \( \pi \) to \( u_2 \).

Therefore, our solution will be defined as follows,

\[
u(x, t) = \begin{cases} 
\pi & x < 0 \\
G \left( \frac{x}{t} \right) & 0 < x < \sin(u_2)t \\
-\frac{\pi}{2} & x > \sin(u_2)t
\end{cases}
\]

where \( G(x/t) = (f')^{-1}(x/t) = \sin^{-1}(x/t) \) where \( \sin(y) \) is restricted to the region \( u_2 \) to \( \pi \).

5. Consider \( f, u^-, u^+ \) shown below.
Consider the initial-value problem
\[
\begin{cases}
    u_t + [f(u)]_x = 0, & t \geq 0 \\
    u(x,0) = \phi(x)
\end{cases}
\]
where
\[
\phi(x) = \begin{cases}
    u^- & x < 0 \\
    u^+ & x > 0.
\end{cases}
\]
Find the weak solution which satisfies the Oleinik entropy condition.

**Answer:** We will use the rubberband method. (See picture below.)

Consider \(u_2\) such that
\[
f'(u_2) = \frac{f(u^-) - f(u_2)}{u^- - u_2}.
\]
From the graph we see that for all \(u \in (u^-, u_2)\),
\[
\frac{f(u^-) - f(u_2)}{u^- - u_2} \leq \frac{f(u^-) - f(u)}{u^- - u}.
\]
Therefore, the Oleinik entropy condition is satisfied for \(f, u^-\) and \(u_2\). In addition, if we put in a curve of discontinuity along the curve \(x = f'(u_2)t\) such that \(u\) jumps from \(u^-\) to \(u_2\), we see that the RH jump condition is satisfied because \(dx/dt = f'(u_2) = [f(u)]/[u]\).

Next, consider \(u_3\) such that
\[
f'(u_3) = \frac{f(u_3) - f(u^+)}{u_3 - u^+}.
\]
We see that for \(u \in (u_3, u^+)\),
\[
\frac{f(u_3) - f(u^+)}{u_3 - u^+} \leq \frac{f(u_3) - f(u)}{u_3 - u}.
\]
Therefore, the Oleinik entropy condition is satisfied for \(f, u_3\) and \(u^+\). In addition, the RH jump condition is satisfied if we define \(x = f'(u_3)t\) as the curve of discontinuity such that \(u\) jumps from \(u_3\) to \(u^+\) along this curve, because \(dx/dt = f'(u_3) = [f(u)]/[u]\). In addition, \(f'\) is strictly increasing in the interval \((u_3, u_2)\). Therefore, \(f'\) is invertible on that interval. Therefore, we can put a rarefaction wave between \(u_3\) and \(u_2\). Therefore, our solution is defined as
\[
u(x,t) = \begin{cases}
    u^- & x < f'(u_2)t \\
    G\left(\frac{x}{f'(u_2)}\right) & f'(u_2)t < x < f'(u_3)t \\
    u^+ & x > f'(u_3)t
\end{cases}
\]
where \(G = (f')^{-1}\) for \(f\) restricted to the interval \((u_2, u_3)\).