1. Classify the following equations as elliptic, parabolic, or hyperbolic.

(a) \(2u_{xx} + 2u_{xy} + 2u_{xz} + 3u_{yy} - 4u_{yz} + 3u_{zz} = 0\)

(b) \(2u_{xx} + u_{yy} = 0\)

(c) \(7u_{xx} - 10u_{xy} - 22u_{yz} + 7u_{yy} - 16u_{xz} - 5u_{zz} = 0\)

Answer:

(a) We have:

\[
\begin{pmatrix}
2 & 1 & 1 \\
1 & 3 & -2 \\
1 & -2 & 3
\end{pmatrix}
\]

The eigenvalues are 0, 3, and 5. \textit{parabolic}

(b) We have:

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\]

The eigenvalues are 1,1,and -1. \textit{hyperbolic}

(c) we have:

\[
\begin{pmatrix}
7 & -5 & -8 \\
-5 & 7 & -11 \\
-8 & -11 & -5
\end{pmatrix}
\]

The eigenvalues are -15.4862, 10.9407, and 13.5455. \textit{hyperbolic}

2. Show that every elliptic equation of the form

\[au_{xx} + bu_{xy} + cu_{yy} + du_x + eu_y + fu = g(x,y),\]

where \(b^2 - 4ac < 0\) can be brought into the form

\[\tilde{u}_{\xi\xi} + \tilde{u}_{\eta\eta} + k\tilde{u} = F(\xi, \eta)\]

through a change of variables. In particular, first give an appropriate linear change of variables to show the equation can be written in the form

\[u_{\xi\xi} + u_{\eta\eta} + \alpha u_\xi + \beta u_\eta + \gamma u = h(\xi, \eta)\]
for some constants $\alpha, \beta, \gamma$. Then introduce a change of variables for the dependent variable $u$ to eliminate the first derivative terms, to show that the equation can be written in the form

$$\ddot{u}_{\xi\xi} + \ddot{u}_{\eta\eta} + k\ddot{u} = F(\xi, \eta).$$

**Answer:** let

$$x = \sqrt{a}\xi; \quad y = \frac{b}{2\sqrt{a}}\xi + \sqrt{(c - b^2/4a)}\eta$$

then we can get that

$$u_{\xi\xi} + u_{\eta\eta} + \alpha u_{\xi} + \beta u_{\eta} + \gamma u = h(\xi, \eta)$$

in which $\alpha$, $\beta$ and $\gamma$ are some suitable number. Then, we let

$$\ddot{u} = ue^{\frac{\alpha\xi + \beta\eta}{2}}$$

and

$$k = \gamma - \frac{\alpha^2 + \beta^2}{4}, \quad F = he^{\frac{\alpha\xi + \beta\eta}{2}}$$

then we get

$$\ddot{u}_{\xi\xi} + \ddot{u}_{\eta\eta} + k\ddot{u} = F(\xi, \eta)$$

3. Reduce the following second-order equation to a system of first-order equations

$$u_{tt} - 4u_{xt} - 5u_{xx} = 0.$$

Then use the method of characteristics to derive the general solution.

**Answer:** Since

$$u_{tt} - 4u_{xt} - 5u_{xx} = (\partial t - 5\partial x)(\partial t + \partial x)u = 0$$

therefore, we let

$$v = (\partial t + \partial x)u$$

hence

$$(\partial t - 5\partial x)v = 0$$

we get

$$v = g(x + 5t)$$

for some function $g$, then let’s try to find our the solution of

$$u_t + u_x = g(x + 5t)$$

then we get

$$\begin{cases}
\frac{dt}{ds} = 1 \\
\frac{dx}{ds} = 1 \\
\frac{du}{ds} = g(x + 5t)
\end{cases}$$
One solution of this system is $t = s, x = 5s$ and $dz/ds = g(10s)$, which implies that

$$z(s) = \frac{1}{10} \int_0^{10s} g(\alpha) d\alpha$$

we arrive at a particular solution

$$u(x, t) = \frac{1}{10} \int_0^{x+5t} h(\alpha) d\alpha = f(x + 5t)$$

We also note that any function of the form $u(x, t) = f(x - t)$ is a solution. So the general solution of the equation is

$$u(x, t) = f(x - t) + g(x + 5t)$$

4. Consider the IVP

$$u_{tt} + u_{xt} - 12u_{xx} = 0$$

\[\begin{align*}
\left\{ \begin{array}{l}
\quad \quad u(x, 0) = \phi(x) \\
\quad \quad u_t(x, 0) = \psi(x)
\end{array} \right. \tag{*}
\]

(a) Make a change of variables to reduce the PDE to canonical form

$$u_{\xi\xi} - u_{\eta\eta} = 0.$$ 

Write the general solution of

$$u_{tt} + u_{xt} - 12u_{xx} = 0.$$ 

**Answer:** Let

$$t = \xi, \quad x = (\xi + 7\eta)/2$$

then we get

$$u_{\xi\xi} - u_{\eta\eta} = 0$$

so the general solution of the PDE is

$$u(x, t) = u(\xi, \eta) = f(\xi + \eta) + g(\xi - \eta) = f((2x + 6t)/7) + g((2x - 8t)/7)$$

(b) Solve the IVP (*).

**Answer:** Use the general solution we get in part (a) and plug in the initial condition, we get

$$\left\{ \begin{array}{l}
\quad \quad f(2x/7) + g(2x/7) = \phi(x) \\
\quad \quad \frac{6}{7}f'(2x/7) - \frac{8}{7}g'(2x/7) = \psi(x)
\end{array} \right.$$

therefore

$$\left\{ \begin{array}{l}
\quad f'(x) = 2\phi'(7x/2) + 1/2\psi'(7x/2) \\
\quad g'(x) = 3/2\phi'(7x/2) - 1/2\psi(7x/2)
\end{array} \right.$$
by change of variables, we get

$$\begin{align*}
\begin{cases}
f((2x + 6t)/7) &= \frac{4}{7} \int_0^{x+3t} \phi'(u) \, du + \frac{1}{7} \int_0^{x+3t} \psi(u) \, du + f(0) \\
g((2x - 8t)/7) &= \frac{3}{7} \int_0^{x-4t} \phi'(u) \, du + \frac{1}{7} \int_{x-4t}^0 \psi(u) \, du + g(0)
\end{cases}
\end{align*}$$

so we get the solution

$$u(x, t) = \frac{4\phi(x + 3t) + 3\phi(x - 4t)}{7} + \frac{1}{7} \int_{x-4t}^{x+3t} \psi(u) \, du$$

5. We say $u$ is a weak solution of the wave equation,

$$\begin{align*}
\begin{cases}
\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} &= 0, \quad -\infty < x < \infty, \quad t > 0 \\
u(x, 0) &= \phi(x) \\
\frac{\partial u}{\partial t}(x, 0) &= \psi(x)
\end{cases}
\end{align*}$$

if

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} u[v_{tt} - v_{xx}] \, dx \, dt + \int_{-\infty}^{\infty} \phi(x) v_t(x, 0) \, dx - \int_{-\infty}^{\infty} \psi(x) v(x, 0) \, dx = 0$$

for all $v \in C^\infty(\mathbb{R} \times [0, \infty))$ with compact support. Let $f$ be a piecewise continuous function with a jump at $y_0$. Show that $u(x, t) = f(x + t)$ is a weak solution of the wave equation. (Note: Similarly it can be shown that a piecewise continuous function of the form $g(x - t)$ is also a weak solution.)

**Answer:** Like we did in class, let

$$\Omega_1 = \{(x, t) : 0 < t < \infty, \quad -\infty < x < y_0 - t\}$$

$$\Omega_2 = \{(x, t) : 0 < t < \infty, \quad y_0 - t < x < \infty\}$$

Then we know

$$\int_0^\infty \int_{-\infty}^{\infty} u(v_{tt} - v_{xx}) \, dx \, dt = \int_{\Omega_1} \int_{\Omega_1} u(v_{tt} - v_{xx}) \, dx \, dt + \int_{\Omega_2} \int_{\Omega_2} u(v_{tt} - v_{xx}) \, dx \, dt$$

and we know that

$$u(v_{tt} - v_{xx}) = (uv_t - u_t v)_t + (u_x v - uv_x)_x + v(u_{tt} - u_{xx})$$

in $\Omega_1$ and $\Omega_2$, $u_{tt} - u_{xx} = 0$, hence

$$\int \int_{\Omega_1} (\ldots) \, dx \, dt = \int \int_{\Omega_1} (u_x v - uv_x) \, dx \, dt + (uv_t - u_t v)_t \, dx \, dt$$

$$= \int_0^\infty (\psi(y_0) v(y_0 - t, t) - f(y_0) v_x(y_0 - t, t)) \, dt$$

$$+ \int_{-\infty}^0 (f(y_0) v_t(x, y_0 - x) - \psi(y_0) v(x, y_0 - x)) \, dx$$

$$- \int_{-\infty}^0 (u(x, 0) v_t(x, 0) - u_t(x, 0) v(x, 0)) \, dx$$
And
\[
\int \int_{\Omega_2} (\ldots) dx dt = \int \int_{\Omega_2} (u_x v - uv_x)_x dx dt + (uv_t - u_t v) dx dt
\]
\[
= \int_{-\infty}^{0} (-\psi(y_0) v(y_0 - t, t) + f(y_0) v_x(y_0 - t, t)) dt
\]
\[
- \int_{-\infty}^{0} (f(y_0) v_t(x, y_0 - x) - \psi(y_0) v(x, y_0 - x)) dx
\]
\[
- \int_{0}^{\infty} (u(x, 0) v_t(x, 0) - u_t(x, 0) v(x, 0)) dx
\]
Add them up, we get
\[
\int_{0}^{\infty} \int_{-\infty}^{\infty} u(v_t - v_x x) dx dt = - \int_{-\infty}^{\infty} \phi(x) v_t(x, 0) dx + \int_{-\infty}^{\infty} \psi(x) v(x, 0) dx
\]