Math 220A - Fall 2002
Homework 5 Solutions

1. Consider the initial-value problem for the hyperbolic equation

$$\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = 0 & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

Use energy methods to show that the domain of dependence of the solution u at the point (x_0, t_0) is the cone $\{(x,t) \in \mathbb{R}^2 : t \geq 0, x_0 - 5(t_0 - t) \leq x \leq x_0 + 4(t_0 - t)\}$.

Answer: For a given function $u = u(x,t)$, define an energy function for this problem as

$$e_u(t) = \int_{x_0-5(t_0-t)}^{x_0+4(t_0-t)} \left[\frac{1}{2}u_t^2 + 10u_x^2\right] dx.$$

(See problem 2 for the motivation behind this derivation.) Now suppose that the initial data is identically zero in the interval $[x_0 - 5t_0, x_0 + 4t_0]$, we will show that $u(x,t) \equiv 0$ for $0 \leq t \leq t_0$, $x \in [x_0 - 5(t_0 - t), x_0 + 4(t_0 - t)]$. Assume u is a solution of (*) such that ϕ and ψ are identically zero in the interval $[x_0 - 5t_0, x_0 + 4t_0]$. Therefore, $e_u(0) = 0$. Further,

$$\begin{aligned}
e_u'(t) &= \int_{x_0-5(t_0-t)}^{x_0+4(t_0-t)} [u_t u_{tt} + 20u_x u_{xt}]\, dx - 4\left[\frac{1}{2}u_t^2 + 10u_x^2\right]_{x=x_0+4(t_0-t)} \\
&\quad - 5\left[\frac{1}{2}u_t^2 + 10u_x^2\right]_{x=x_0-5(t_0-t)} \\
&= \int_{x_0-5(t_0-t)}^{x_0+4(t_0-t)} [u_t u_{tt} - 20u_{xx}u_t]\, dx + 20u_x u_t\Big|_{x=x_0-5(t_0-t)}^{x=x_0+4(t_0-t)} \\
&\quad - 4\left[\frac{1}{2}u_t^2 + 10u_x^2\right]_{x=x_0+4(t_0-t)} - 5\left[\frac{1}{2}u_t^2 + 10u_x^2\right]_{x=x_0-5(t_0-t)} \\
&= -\int_{x_0-5(t_0-t)}^{x_0+4(t_0-t)} u_t u_{xt}\, dx - 2\left[u_t^2 - 10u_x u_t + 20u_x^2\right]_{x=x_0+4(t_0-t)} \\
&\quad - \frac{5}{2}\left[u_t^2 + 8u_x u_t + 20u_x^2\right]_{x=x_0-5(t_0-t)} \\
&= -\frac{5}{2}\left[u_t^2 - 8u_x u_t + 16u_x^2\right]_{x=x_0+4(t_0-t)} \\
&\quad - 2\left[u_t^2 + 10u_x u_t + 25u_x^2\right]_{x=x_0-5(t_0-t)} \\
&= -\frac{5}{2}[u_t - 4u_x]^2|_{x=x_0+4(t_0-t)} - 2[u_t + 5u_x]^2|_{x=x_0-5(t_0-t)} \\
&\leq 0.
\end{aligned}$$

Therefore, we conclude that $e_u(t) \leq 0$. But, by definition, $e_u(t) \geq 0$. Therefore, we conclude that $e_u(t) \equiv 0$. Therefore, $u_t \equiv 0 \equiv u_x$ inside this cone. This implies that u

is constant in the cone. But, u is assumed to be identically zero at $t = 0$. Therefore, we conclude that $u \equiv 0$ inside the cone. Therefore, the solution depends at most on the value of the initial data in the interval $[x_0 - 5(t_0 - t), x_0 + 4(t_0 - t)]$. (*Remark:* By solving the equation, it can be shown that the solution depends on the values of the initial data in this entire interval.)

2. Use energy methods to prove uniqueness of solutions to

$$\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = f(x,t) & -\infty < x < \infty, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

assuming that ϕ and ψ have compact support.

Answer: To find an energy function associated with this PDE, we multiply the homogeneous equation by u_t and integrate over $\mathbb{R}$,

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} u_t(u_{tt} + u_{xt} - 20u_{xx})\,dx \\ &= \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2)_t + \frac{1}{2}(u_t^2)_x + 20u_{xt}u_x\,dx - 20u_t u_x|_{x\to-\infty}^{x\to+\infty} \\ &= \int_{-\infty}^{\infty} \frac{1}{2}(u_t^2)_t + 10(u_x^2)_t\,dx + \left[\frac{1}{2}u_t^2 - 20u_t u_x\right]\Bigg|_{x\to-\infty}^{x\to+\infty} \\ &= \frac{\partial}{\partial t}\left(\int_{-\infty}^{\infty} \frac{1}{2}u_t^2 + 10u_x^2\,dx\right), \end{aligned}$$

assuming u has compact support. Therefore, we define an energy for this problem as follows. For a given function $u = u(x,t)$, we let

$$E_u(t) = \int_{-\infty}^{\infty} \frac{1}{2}u_t^2 + 10u_x^2\,dx.$$

Now suppose there are two solutions u, v of our initial-value problem above. Let $w = u - v$. Then w is a solution of

$$\begin{cases} w_{tt} + w_{xt} - 20w_{xx} = 0 & -\infty < x < \infty, t > 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0 \end{cases}$$

By the method of derivation of the energy function above, we see that $E_w'(t) = 0$ (using the fact that the initial data has compact support, and, therefore, the solution has compact support). But, the initial data is identically zero. Therefore, $E_w(0) = 0$. We conclude that $E_w(t) \equiv 0$. Therefore, using the fact that the integrand is non-negative, we conclude that $w_t(x,t) \equiv 0 \equiv w_x(x,t)$. But, this implies that $w \equiv$ const. Using the fact that $w(x,0) \equiv 0$, we conclude that $w(x,t) \equiv 0$. Therefore, $u \equiv v$.

3. Consider the initial-value problem for the following hyperbolic equation,

$$\begin{cases} ru_{tt} - \nabla \cdot (p\nabla u) + qu = F & x \in \mathbb{R}^n, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

where $r(x), p(x)$ are positive and $q(x)$ is non-negative. Use energy methods to prove uniqueness of solutions to this problem.

Answer: First, we derive an energy associated with this PDE. Multiplying the homogeneous equation by u_t and integrating over $\mathbb{R}^n$, we have

$$\begin{aligned} 0 &= \int_{\mathbb{R}^n} u_t(ru_{tt} - \nabla \cdot (p\nabla u) + qu)\,dx \\ &= \int_{\mathbb{R}^n} r\frac{1}{2}(u_t^2)_t + p\nabla u_t \cdot \nabla u + q\frac{1}{2}(u^2)_t\,dx - \int_{\partial\Omega} pu_t \nabla u \cdot n\,dS(x), \end{aligned}$$

where Ω is the support of u. If the initial data has compact support, then the solution will have compact support (because this equation is hyperbolic). Therefore, assuming u vanishes as $|x| \to +\infty$, we conclude that

$$0 = \int_{\mathbb{R}^n} r\frac{1}{2}(u_t^2)_t + \frac{1}{2}p(|\nabla u|^2)_t + \frac{1}{2}q(u^2)_t\,dx.$$

For a given function $u = u(x,t)$, let

$$E_u(t) = \frac{1}{2}\int_{\mathbb{R}^n} ru_t^2 + p|\nabla u|^2 + qu^2\,dx.$$

By the method of derivation above, we see that if u is a solution of the homogeneous PDE above, and u vanishes as $|x| \to +\infty$, we know that $E_u'(t) = 0$.

Now suppose u and v are both solutions of the inhomogeneous problem stated above. Let $w = u - v$. Then w is a solution of

$$\begin{cases} rw_{tt} - \nabla \cdot (p\nabla w) + qw = 0 & x \in \mathbb{R}^n, t > 0 \\ w(x,0) = 0 \\ w_t(x,0) = 0. \end{cases}$$

Therefore, $E_w'(t) = 0$. In addition, the initial data is identically zero. Therefore, $E_w(0) = 0$. We conclude that $E_w(t) \equiv 0$. Therefore, we have $w \equiv 0$ which implies $u \equiv v$.

4. Use Duhamel's principle to derive formulas for the solutions of the following initial value problems.

   (a)

$$\begin{cases} u_t + au_x = f(x,t) \\ u(x,0) = \phi(x) \end{cases}$$

i. First find the solution operator $S(t)$ associated with the homogeneous equation.
Answer : The solution to the homogeneous equation is:

$$\begin{aligned} u_{hom}(x,t) &= S(t)\phi(x) \\ &= \phi(x-at) \end{aligned}$$

ii. Use $S(t)$ to derive the solution of the inhomogeneous equation.
Answer : We get $S(t)$ from part (i), so by Duhamel's principle,

$$\begin{aligned} u(x,t) &= S(t)\phi(x) + \int_0^t S(t-s)f(x,s)ds \\ &= \phi(x-at) + \int_0^t f(x-a(t-s),s)ds \end{aligned}$$

(b)

$$(*)\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = f(x,t) \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

i. Write the equation as a system

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi \end{cases}$$

Answer : Let $u_t = v$, we can write the inhomegeneous wave equation as:

$$\begin{cases} u_t = v; \\ v_t = 20u_{xx} - u_{xt} + f(x,t) \end{cases}$$

which can be written in a matrix form as:

$$\begin{bmatrix} u \\ v \end{bmatrix}_t = \begin{bmatrix} 0 & 1 \\ 20\partial_x^2 & -\partial_x \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} 0 \\ f(x,t) \end{bmatrix}$$

let

$$U = \begin{bmatrix} u \\ v \end{bmatrix} \; A = \begin{bmatrix} 0 & 1 \\ -20\partial_x^2 & \partial_x \end{bmatrix} \; F = \begin{bmatrix} 0 \\ f(x,t) \end{bmatrix} \; \Phi = \begin{bmatrix} \phi(x) \\ \psi(x) \end{bmatrix}$$

Our equation can be written in matrix form as:

$$\begin{cases} U_t + AU = F \\ U(0) = \Phi \end{cases}$$

ii. Find the solution operator $S(t)$ associated with the homogeneous system

$$\begin{cases} U_t + AU = 0 \\ U(0) = \Phi = \begin{bmatrix} \phi \\ \psi \end{bmatrix}. \end{cases}$$

Answer : Consider

$$\begin{cases} U_t + AU = 0 \\ U(x,t) = \Phi \end{cases}$$

then this is the equation:

$$\begin{cases} u_{tt} + u_{xt} - 20u_{xx} = 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x) \end{cases}$$

We know the solution to the homogeneous equation is:

$$u(x,t) = \frac{5\phi(x+4t) + 4\phi(x-5t)}{9} + \frac{1}{9}\int_{x-5t}^{x+4t} \psi(y)dy$$

The solution of the homogeneous equation is given by

$$U(x,t) = \left[\begin{array}{l} \frac{1}{9}(5\phi(x+4t) + 4\phi(x-5t)) + \frac{1}{9}\int_{x-5t}^{x+4t} \psi(y)dy \\ \frac{20}{9}(\phi^{'}(x+4t) - \phi^{'}(x-5t)) + \frac{1}{9}(4\psi(x+4t) + 5\psi(x-5t)) \end{array} \right]$$

In other words, defining the solution operator $S(t)$ as

$$S(t)\Phi = S(t)\left[\begin{array}{c} \phi \\ \psi \end{array} \right] = \left[\begin{array}{l} \frac{1}{9}(5\phi(x+4t) + 4\phi(x-5t)) + \frac{1}{9}\int_{x-5t}^{x+4t} \psi(y)dy \\ \frac{20}{9}(\phi^{'}(x+4t) - \phi^{'}(x-5t)) + \frac{1}{9}(4\psi(x+4t) + 5\psi(x-5t)) \end{array} \right]$$

iii. Use the solution operator $S(t)$ to find the solution of the inhomogeneous system, and use this to find the solution of (*).
Answer : We have found the operator $S(t)$ from above. Use Duhamel's principle, as to the in homogeneous problem:

$$U(x,t) = S(t)\Phi + \int_0^t S(t-s)F(s)ds$$

Looking at the first component of this vector-valued equation (6), we see this would imply that:

$$u(x,t) = \frac{1}{9}(5\phi(x+4t)+4\phi(x-5t)) + \frac{1}{9}\int_{x-5t}^{x+4t} \psi(y)dy + \int_0^t \frac{1}{9}\int_{x-5(t-s)}^{x+4(t-s)} f(y,s)dyds.$$

5. Use Green's Theorem to derive the solution of the inhomogeneous wave equation on the half-line,

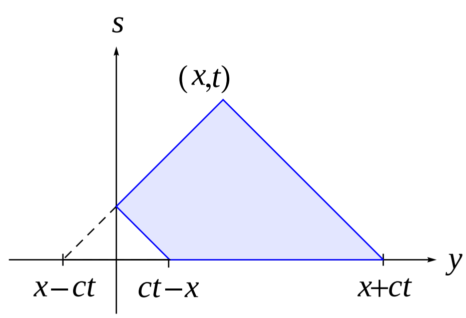
$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) & 0 < x < \infty \\ u(x,0) = \phi(x) & 0 < x < \infty \\ u_t(x,0) = \psi(x) & 0 < x < \infty \\ u(0,t) = h(t), & \end{cases}$$

where we assume $\phi(0) = \psi(0) = h(0) = 0$.

Answer: When $x - ct \geq 0$, the solution's domain of dependence does not interact with the $x = 0$ axis. Thus the solution is the usual solution to the inhomogeneous wave equation:

$$u(x,t) = \frac{1}{2}(\phi(x+ct) + \phi(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds$$

For $x - ct < 0$, we integrate over the region shown



We have by Green's theorem:

$$\int\int_{\Omega} f(y,s)dyds = \int\int_{\Omega}(u_{ss} - c^2 u_{yy})dyds = -\int_{\partial\Omega}(c^2 u_y ds + u_s dy) = -(I + J + K + L)$$

where:

$$I = \int_{L_1}(c^2 u_y ds + u_s dy) = \int_{L_1}(u_s dy) = \int_{ct-x}^{x+ct} \psi(y)dy$$

$$J = \int_{L_2}(c^2 u_y ds + u_s dy) = \int_{L_2}(-cu_x dx + u_t(-cdt)) = -cu(x,t) + c\phi(x+ct)$$

$$K = \int_{L_3}(c^2 u_y ds + u_s dy) = \int_{L_3}(cu_y dy + cu_s(cds)) = -c(h(t - \frac{x}{c}) - cu(x,t))$$

$$L = \int_{L_4}(c^2 u_y ds + u_s dy) = \int_{L_4}(-cu_y dy + u_s(cds)) = -c\phi(ct-x) + ch(t - \frac{x}{c})$$

Thus

$$u(x,t) = \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + h(t - \frac{x}{c}) + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy + \frac{1}{2c}\iint_{\Omega} f(y,s)dyds$$