Math 220a - Fall 2002 Homework 6 Solutions

1. Use the method of reflection to solve the initial-boundary value problem on the interval $0 < x < l$,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 & 0 < x < l \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = x & 0 < x < l \\ u(0,t) = 0 = u(l,t). & \end{cases}$$

In particular, calculate the explicit solution of u in regions R_1, R_2, R_3 shown below.

Solution:

$$u(x,t) = \begin{cases} xt, & (x,t) \in R_1 \\ \frac{1}{c}l^2 - \frac{1}{c}l(x+ct) + xt, & (x,t) \in R_2 \\ xt - \frac{1}{c}l(x+ct) + \frac{1}{c}l^2, & (x,t) \in R_3 \end{cases}$$

2. Do the same thing as in problem 1, except now for the Neumann boundary conditions. That is, use the method of refelection to solve the inital-boundary value problem on the interval $0 < x < l$ with Neumann boundary conditions,

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & 0 < x < l \\ u(x,0) = 0 & 0 < x < l \\ u_t(x,0) = x & 0 < x < l \\ u_x(0,t) = 0 = u_x(l,t). & \end{cases}$$

Write the explicit solution in the same three regions as shown in problem 1.

Solution:

a. For $x \in R_1$

$$\begin{aligned} u(x,t) &= \frac{1}{2c}\left(\int_{x-ct}^{x+ct} \phi_{even} dx\right) \\ &= \frac{1}{2c}\left(\int_{x-ct}^{0} \phi_{even} dx + \int_{0}^{x+ct} \phi_{even} dx\right) \\ &= \frac{1}{2c}\left(\int_{0}^{ct-x} x dx + \int_{0}^{x+ct} x dx\right) \\ &= \frac{(ct)^2 + x^2}{2c} \end{aligned}$$

b. For $x \in R_2$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\left(\int_{x-ct}^{x+ct} \phi_{even} dx\right) \\
&= \frac{1}{2c}\left(\int_{x-ct}^{0} \phi_{even} dx + \int_{0}^{l} \phi_{even} dx + \int_{l}^{x+ct} \phi_{even} dx\right) \\
&= \frac{1}{2c}\left(\int_{0}^{ct-x} x dx + \int_{0}^{l} x dx + \int_{2l-x-ct}^{l} x dx\right) \\
&= \frac{-l^2 - 2ctx + 2l(ct+x)}{2c}
\end{aligned}
$$

c. For $x \in R_3$

$$
\begin{aligned}
u(x,t) &= \frac{1}{2c}\left(\int_{x-ct}^{x+ct} \phi_{even} dx\right) \\
&= \frac{1}{2c}\left(\int_{x-ct}^{0} \phi_{even} dx + \int_{0}^{l} \phi_{even} dx + \int_{l}^{2l} \phi_{even} dx + \int_{2l}^{x+ct} \phi_{even} dx\right) \\
&= \frac{1}{2c}\left(\int_{0}^{ct-x} x dx + 2\int_{0}^{l} x dx + \int_{0}^{x+ct-2l} x dx\right) \\
&= \frac{2l^2 + (x-ct)^2 + (x+ct-2l)^2}{4c}
\end{aligned}
$$

3. Use Duhamel's principle to find the solution of the *inhomogeneous* wave equation on the half-line with Neumann boundary conditions

$$
\begin{cases}
u_{tt} - c^2 u_{xx} = f(x,t), & 0 < x < \infty \\
u(x,0) = \phi(x) & 0 < x < \infty \\
u_t(x,0) = \psi(x) & 0 < x < \infty \\
u_x(0,t) = 0. &
\end{cases}
$$

In particular, introducing a new function $v = u_t$, rewrite the equation as the system

$$
\begin{cases}
U_t + AU = F & 0 < x < \infty \\
U(x,0) = \Phi(x) & 0 < x < \infty \\
U_x(0,t) = \begin{bmatrix} u_x(0,t) \\ v_x(0,t) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} &
\end{cases}
$$

where

$$
\begin{aligned}
U &= \begin{bmatrix} u \\ v \end{bmatrix} \qquad & A &= \begin{bmatrix} 0 & -1 \\ -c^2\partial_x^2 & 0 \end{bmatrix} \\
F &= \begin{bmatrix} 0 \\ f \end{bmatrix} \qquad & \Phi &= \begin{bmatrix} \phi \\ \psi \end{bmatrix}.
\end{aligned}
$$

(a) Find the solution operator $S(t)$ associated with the homogeneous system

$$\begin{cases} U_t + AU = 0 & 0 < x < \infty \\ U(x,0) = \Phi(x) & 0 < x < \infty \\ U_x(0,t) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{cases}$$

Solution: Extending the initial data ϕ and ψ to be even, we know that the solution of the wave equation on the half-line with Neumann boundary conditions is given as follows:

$$u(x,t) = \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy.$$

In particular, for $x > ct$, we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy.$$

While for $x < ct$, we have

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(ct-x)] + \frac{1}{2c}\int_{0}^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_{0}^{ct-x} \psi(y)\,dy.$$

Therefore, the solution operator associated with the system above is given by

$$\boxed{S(t)\Phi = \begin{bmatrix} \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy \\ \frac{\partial}{\partial t}\left(\frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy\right) \end{bmatrix}.}$$

(b) Use $S(t)$ to construct a solution of the inhomogeneous system.

Solution:

By Duhamel's principle the solution of the inhomogeneous system will be given by

$$S(t)\Phi + \int_0^t S(t-s)F(s)\,ds.$$

Therefore, the solution of the inhomogeneous system is given by

$$\begin{bmatrix} \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy \\ \frac{\partial}{\partial t}\left(\frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy\right) \end{bmatrix}$$
$$+ \int_0^t \begin{bmatrix} \frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f_{even}(y,s)\,dy \\ \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{x-c(t-s)}^{x+c(t-s)} f_{even}(y,s)\,dy\right) \end{bmatrix} ds.$$

(c) Use the solution of the inhomogeneous system to solve the inhomogeneous wave equation on the half-line with Neumann boundary conditions.

Solution: Therefore, the solution of the inhomogeneous wave equation on the half-line with Neumann boundary conditions is given by the first component of the vector-valued function found in part (b),

$$\begin{aligned} u(x,t) = \frac{1}{2}[\phi_{even}(x+ct) + \phi_{even}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{even}(y)\,dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{even}(y,s) dy ds \end{aligned}$$

In particular, for $x > ct$,

$$\begin{aligned} u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)\,dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s) dy ds. \end{aligned}$$

while for $x < ct$, defining t_0 such that $x - c(t-t_0) = 0$,

$$\begin{aligned} u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct} \psi(y)\,dy + \frac{1}{2c}\int_0^{ct-x} \psi(y)\,dy \\ + \frac{1}{2c}\int_0^{t_0} \int_0^{x+c(t-s)} f(y,s)\,dy\,ds + \frac{1}{2c}\int_0^{t_0} \int_0^{c(t-s)-x} f(y,s)\,dy\,ds \\ + \frac{1}{2c}\int_{t_0}^{t} \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)\,dy\,ds. \end{aligned}$$

4. Use separation of variables to solve

$$\begin{aligned} u_{tt} - c^2 u_{xx} = 0 \quad & 0 < x < l, t > 0 \\ u(x,0) = x(x-l)^2 \quad & 0 < x < l \\ u_t(x,0) = 0 \quad & 0 < x < l \\ u(0,t) = u_x(l,t) = 0 \quad & \end{aligned}$$

Solution:

letting $u(x,t) = X(x)T(x)$ we have,

$$\frac{T''}{c^2 T} = \frac{X''}{X} = \lambda$$

Therefore $X(x)$ is of the form,

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

boundary conditions imply $X(0) = 0$ and $X'(l) = 0$ which yields that

$$C = 0$$

and

$$\beta = \frac{(2n+1)\pi}{2l}$$

Therefore

$$X(x) = \sin\left(\frac{(2n+1)\pi}{2l}x\right)$$

Now $T(t)$ satisfies

$$T'' = c^2\lambda T$$

Therefore for each λ_n, T takes the form,

$$T(t) = A_n\cos\left(\frac{(2n+1)\pi c}{2l}t\right) + B_n\sin\left(\frac{(2n+1)\pi c}{2l}t\right)$$

Using the boundary condition $T'(0) = 0$ we have $B_n = 0$ So we have for $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{(2n+1)\pi c}{2l}t\right)\sin\left(\frac{(2n+1)\pi}{2l}x\right)\right]$$

using the boundary conditions for $u(x,0)$ and orthogonality arguments, we have for A_n

$$A_n = \frac{2}{l}\int\sin\left(\frac{(2n+1)\pi}{2l}x\right)x(x-l)^2dx$$

5. Consider the eigenvalue problem,

$$\begin{aligned} &-X'' = \lambda X && 0 < x < 1 \\ &X'(0) + aX(0) = 0 \\ &X(1) = 0 \end{aligned}$$

Solution: When we are looking for positive eigenvalues, we can take $\lambda = \beta^2$. We are then looking for solutions to

$$X'' + \beta^2X = 0$$

Solutions to the above are of the form

$$X = C\cos(\beta x) + D\sin(\beta x)$$

Now using the boundary conditions $X'(0) + aX(0) = 0$,

$$D\beta + aC = 0$$

and

$$C\cos(\beta) + D\sin(\beta) = 0$$

Simplyfing yields,

$$\tan(\beta) = \frac{\beta}{a}$$

Graphically, we can show that the above has infinite number of solutions by plotting both, $\tan(\beta)$ and $\frac{\beta}{a}$ for all values of β. One would see that the graphs of both the functions intersect infinite number of times, thereby indicating infinite number of solutions.

Next, when we are looking for negative eigne values, we use $\lambda = -\beta^2$ and follow similar arguments as above and arrive at the final equation for β as,

$$\tanh(\beta) = \frac{\beta}{a}$$

One can see by plotting the graphs of $\tanh(\beta)$ and $\frac{\beta}{a}$, that when $a \leq 1$ there is no intersection and when $a > 1$ there is exactly 1 intersection.

6. Use seperation of variables to solve

$$\begin{aligned} &u_{tt} - c^2 u_{xx} + \gamma^2 u = 0 \qquad\qquad 0 < x < l, t > 0 \\ &u(x,0) = \phi(x) \\ &u_t(x,0) = \psi(x) \\ &u(0,t) = u(l,t) = 0 \end{aligned}$$

where $\gamma > 0$

Solution:

letting $u(x,t) = X(x)T(x)$ we have,

$$\frac{T''}{c^2 T} = \frac{X''}{X} - \frac{\gamma^2}{c^2} = \lambda$$

Therefore $X(x)$ solves $X'' = (\lambda + \frac{\gamma^2}{c^2})X$ is of the form,

$$X(x) = C\cos(\beta x) + D\sin(\beta x)$$

boundary conditions imply $X(0) = 0$ and $X(l) = 0$ which yields that

$$C = 0$$

and

$$\beta = \frac{n\pi}{l}$$

with

$$\lambda_n = \frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}$$

Now $T(t)$ satisfies

$$T'' = c^2\lambda T$$

Therefore for each λ_n, T takes the form,

$$T(t) = A_n \cos\left(\sqrt{\frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}}ct\right) + B_n \sin\left(\sqrt{\frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}}ct\right)$$

So we have for $u(x,t)$

$$u(x,t) = \sum_{n=1}^{\infty}\left[A_n \cos\left(\sqrt{\frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}}ct\right) + B_n \sin\left(\sqrt{\frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}}ct\right)\right]\sin\left(\frac{n\pi}{l}x\right)$$

using the boundary conditions for $u(x,0)$ and orthogonality arguments, we have for A_n and B_n

$$A_n = \frac{2}{l}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\phi(x)dx$$

and

$$B_n = \frac{2}{lc}\left(\frac{n^2\pi^2}{l^2} - \frac{\gamma^2}{c^2}\right)^{-1/2}\int_0^l \sin\left(\frac{n\pi}{l}x\right)\psi(x)dx$$