Math 220A - Fall 2002
Homework 7 Solutions

1. Suppose that the series

$$\sum_{n=1}^{\infty} f_n(x)$$

converges uniformly on $[a, b]$ to $f(x)$. Show that

$$\sum_{n=1}^{\infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx$$

provided f_n and f are integrable on $[a, b]$.

Answer: Since f_n and f are integrable, the partial sum

$$S_N = \sum_{n=1}^{N} \int_a^b f_n(x)\ dx$$

and

$$I = \int_a^b f(x)\ dx$$

are defined. Subtracting and using the linearity if integration we have

$$I - S_N = \int_a^b f(x) - \sum_{n=1}^{N} f_n(x)\ dx$$

Since the series converges uniformly to f on $[a, b]$, given $\epsilon > 0$ there exists some N_0 such that

$$\left| f(x) - \sum_{n=1}^{N} f_n(x) \right| < \epsilon$$

for $N \geq N_0$ and all $x \in [a, b]$. Then

$$|I - S_N| \leq (b - a)\epsilon$$

Thus $S_N \to I$.

2. Suppose that f is in $C^2([-L, L])$ and satisfies $f(L) = f(-L)$.

   (a) Show that, for some positive constant C, the Fourier coefficients satisfy

$$|A_n| \leq \frac{C}{n^2} \qquad \text{and} \qquad |B_n| \leq \frac{C}{n^2}$$

   for all integers $n \geq 1$.

Answer: Consider B_n - the proof for A_n is the same. Integrate by parts to get

$$\begin{aligned} B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx \\ &= \frac{-\cos\left(\frac{n\pi x}{L}\right)f(x)}{n\pi}\Bigg|_{-L}^{L} + \frac{1}{n\pi}\int_{-L}^{L} f'(x)\cos\left(\frac{n\pi x}{L}\right)\,dx \end{aligned}$$

Since $f(-L) = f(L)$, the first term vanishes. Integrating by parts again we get

$$\frac{L\sin\left(\frac{n\pi x}{L}\right)f'(x)}{n^2\pi^2}\Bigg|_{-L}^{L} - \frac{L}{n^2\pi^2}\int_{-L}^{L} f''(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

Again the first term vanishes since $\sin\left(\frac{n\pi x}{L}\right)$ equals zero at the endpoints (in the calculation of A_n this term will not vanish, but will be some constant times $1/n^2$, as desired). Since f is C^2 the integrand is a continuous function, bounded independently of n, and thus the integral is also bounded by a constant independent of n. This proves the assertion.

(b) Show that the Fourier series for f converges absolutely at each point x in $[-L, L]$. Note: You do not need to prove convergence to $f(x)$.

Answer: Bounding the trig functions by 1, and using part (a), we have

$$\sum_{n=1}^{\infty}\left|A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right| \le C\sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is a convergent series.

3. (a) Calculate the Fourier sine series for $f(x) = \cos(x)$ on the interval $[0, \pi]$.

Answer: Suppose $f(x) = \sum_{n=1}^{\infty} a_n \sin(nx)$, then we have

$$a_n = \frac{\int_0^{\pi}\sin(nx)\cos(x)dx}{\int_0^{\pi}\sin^2(nx)dx}$$

Hence,

$$a_n = -\frac{1}{\pi}\{\frac{(-1)^{(n+1)}-1}{n+1} + \frac{(-1)^{(n-1)}-1}{n-1}\}$$

i.e

$$\begin{cases} a_n = 0; \quad n = 2k+1 \\ a_n = \dfrac{4n}{\pi(n^2-1)}; \quad n = 2k \end{cases}$$

hence

$$\cos(x) = \sum_{k=1}^{\infty}\frac{8k}{\pi(4k^2-1)}\sin(2kx)$$

(b) Justify that the Fourier sine series obtained in part (a) converges to $\cos(x)$ pointwise on $(0, \pi)$ but not uniformly on $[0, \pi]$.

Answer: Use Theorem 7 in class notes, we know that the Fourier sine series converges to $\cos(x)$ pointwise on $(0, \pi)$. But we see that the series equals zero at both end point (0 and π) but the function $\cos(x)$ is not 0 there. Hence the series converge pointwise but not uniformly on $[0, \pi]$.

(c) Show that term-by-term differentiation fails.

Answer: Since we know that the derivative of $\cos(x)$ is $-\sin(x)$. Suppose we can differentiate the series term by term, then we get

$$\sum_{k=1}^{\infty} \frac{16k^2}{\pi(4k^2-1)} \cos(2kx)$$

This is clearly divergent because the terms don't tend to zero as n goes to ∞ (the nth term test for divergence). So we can not differentiate term by term.

4. Solve

$$\begin{cases} u_{tt} - c^2 u_{xx} = e^t \sin(5x) & 0 < x < \pi \\ u(x,0) = 0 \\ u_t(x,0) = \sin(3x) \\ u(0,t) = 0 = u(\pi, t) \end{cases}$$

Answer: We know that the solution of the homogeneous equation

$$\begin{cases} v_{tt} - c^2 v_{xx} = 0 & 0 < x < \pi \\ v(x,0) = 0 \\ v_t(x,0) = \sin(3x) \\ v(0,t) = 0 = v(\pi, t) \end{cases}$$

is given by

$$v(x,t) = \sum_{n=1}^{\infty} A_n \sin(cnt) \sin(nx)$$

where

$$ncA_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) \sin(3x) dx$$

hence we get $A_3 = 1/3c$ and $A_n = 0$ for other ns. Similarly like what we did in our notes, as to the inhomogeneous equation, we get

$$C_n(s) = 0$$

and

$$ncD_n(s) = \frac{2}{\pi}\int_0^\pi \sin(nx)e^s \sin(5x)dx$$

hence, $D_5(s) = e^s/5c$ and $D_n(s) = 0$ for other ns. Therefore, the solution is given by

$$u(x,t) = \frac{1}{3c}\sin(3ct)\sin(3x) + \int_0^t \frac{e^s \sin(5c(t-s))\sin(5x)}{5c}ds$$

5. Solve

$$\begin{cases} u_{tt} - u_{xx} = 0 & 0 < x < \pi \\ u(x,0) = 0 \\ u_t(x,0) = 0 \\ u(0,t) = g(t) \\ u_x(\pi,t) + u(\pi,t) = h(t) \end{cases}$$

Answer: Use the Method of Shifting the Data, introduce a new function $W(x,t)$ such that

$$\begin{cases} W(0,t) = g(t) \\ W_x(\pi,t) + W(\pi,t) = h(t) \end{cases}$$

then we get

$$W(x,t) = (1 - \frac{x}{\pi+1})g(t) + \frac{x}{\pi+1}h(t)$$

then defining $v(x,t) = u(x,t) - W(x,t)$, we will get the new intial-value problem which now has zero boundary data,

$$\begin{cases} v_{tt} - v_{xx} = -(1-\frac{x}{\pi+1})g''(t) - \frac{x}{\pi+1}h''(t) = -W_{tt}(x,t) & 0 < x < \pi \\ v(x,0) = -(1-\frac{x}{\pi+1})g(0) - \frac{x}{\pi+1}h(0) = -W(x,0) \\ v_t(x,0) = -(1-\frac{x}{\pi+1})g'(0) - \frac{x}{\pi+1}h'(0) = -W_t(x,0) \\ v(0,t) = 0 \\ v_x(\pi,t) + v(\pi,t) = 0 \end{cases}$$

First let's solve the homogeneous equation

$$\begin{cases} U_{tt} - U_{xx} = 0 & 0 < x < \pi \\ U(x,0) = -(1-\frac{x}{\pi+1})g(0) - \frac{x}{\pi+1}h(0) = -W(x,0) \\ U_t(x,0) = -(1-\frac{x}{\pi+1})g'(0) - \frac{x}{\pi+1}h'(0) = -W_t(x,0) \\ U(0,t) = 0 \\ U_x(\pi,t) + U(\pi,t) = 0 \end{cases}$$

then we get

$$U(x,t) = \sum_{n=1}^\infty (a_n \cos(\beta_n t) + b_n \sin(\beta_n t))\sin(\beta_n x)$$

where β_n satisfy

$$\beta_n = -\tan(\beta_n \pi)$$

and a_n satisfy

$$a_n = \frac{\int_0^\pi (-W(x,0)\sin(\beta_n x))dx}{\int_0^\pi \sin^2(\beta_n x)dx}$$

and b_n satisfy

$$\beta_n b_n = \frac{\int_0^\pi (-W_t(x,0)\sin(\beta_n x))dx}{\int_0^\pi \sin^2(\beta_n x)dx}$$

As to the inhomogeneous problem, we get

$$\beta_n D_n(s) = \frac{\int_0^\pi (-W(x,s)\sin(\beta_n x))dx}{\int_0^\pi \sin^2(\beta_n x)dx}$$

then, the solution is

$$v(x,t) = U(x,t) + \int_0^t \sum_{n=1}^{\infty} D_n(s)\sin(\beta_n(t-s))\sin(\beta_n x)ds$$

hence, our solution is

$$u(x,t) = v(x,t) + W(x,t)$$