Math 220A - Fall 2002
Homework 8 Solutions

1. Consider

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & x \in \mathbb{R}^3, t > 0 \\ u(x,0) = \phi(x) \\ u_t(x,0) = \psi(x). \end{cases}$$

   Suppose ϕ, ψ are supported in the annular region $a < |x| < b$.

   (a) Find the time $T_1 > 0$ such that $u(x,t)$ is definitely zero for $t > T_1$ in the case when

      i. $|x| > b$
      Answer:
      $$\boxed{T_1 = \frac{|x| + b}{c}.}$$

      ii. $a < |x| < b$
      Answer:
      $$\boxed{T_1 = \frac{|x| + b}{c}.}$$

      iii. $|x| < a$.
      Answer:
      $$\boxed{T_1 = \frac{|x| + b}{c}.}$$

   (b) Find the time $T_2 > 0$ such that $u(x,t)$ is definitely zero for $0 < t < T_2$ in the case when

      i. $|x| > b$
      Answer:
      $$\boxed{T_2 = \frac{|x| - b}{c}.}$$

      ii. $|x| < a$.
      Answer:
      $$\boxed{T_2 = \frac{a - |x|}{c}.}$$

   (c) Consider the same questions for $n = 2$ dimensions.
   Answer: Since $u(x,t)$ depends on the values of the initial data in $B(x,ct)$, there is no time T_1 such that we can guarantee that $u(x,t) \equiv 0$ for all $t > T_1$. In answer to part (b), we again have $\boxed{T_2 = (|x| - b)/c \text{ if } |x| > b}$ and $\boxed{T_2 = (a - |x|)/c \text{ if } |x| < a}$.

2. Solve

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & (x,y,z) \in \mathbb{R}^3, t > 0 \\ u(x,y,z,0) = 1 \\ u_t(x,y,z,0) = x^2 + y^2 + z^2. \end{cases}$$

Answer: *Note: Below x, y and z represent vectors in* $\mathbb{R}^3$*.* Our solution is given by Kirchoff's formula as

$$\begin{aligned} u(x,t) &= ⨍_{\partial B(x,ct)} \phi(y) + \nabla\phi(y) \cdot (y-x) + t\psi(y)\, dS(y) \\ &= ⨍_{\partial B(x,ct)} 1 + t|y|^2\, dS(y). \end{aligned}$$

By making a change of variables $y = x + ctz$, we can rewrite this as

$$\begin{aligned} u(x,t) &= ⨍_{\partial B(0,1)} [1 + t|x+ctz|^2]\, dS(z) \\ &= ⨍_{\partial B(0,1)} [1 + t|x|^2 + 2ct^2 x \cdot z + c^2t^3|z|^2]\, dS(z). \end{aligned}$$

Now

$$⨍_{\partial B(0,1)} 1\, dS(z) = 1$$

$$⨍_{\partial B(0,1)} t|x|^2\, dS(z) = t|x|^2$$

$$⨍_{\partial B(0,1)} 2ct^2 x \cdot z\, dS(z) = 0$$

$$⨍_{\partial B(0,1)} c^2t^3|z|^2\, dS(z) = c^2t^3.$$

Therefore, we conclude that our solution is

$$\boxed{u(x,t) = 1 + t|x|^2 + c^2t^3.}$$

3. Solve

$$\begin{cases} u_{tt} - c^2 \Delta u = 0 & (x,y) \in \mathbb{R}^2, t > 0 \\ u(x,y,0) = 0 \\ u_t(x,y,0) = x^2 + y^2 \end{cases}$$

Answer: *Note: Below, x, y and z represent vectors in* $\mathbb{R}^2$*.* The solution is given by the formula

$$\begin{aligned} u(x,t) &= \frac{1}{2\pi c^2t^2} \int_{B(x,ct)} \frac{ct\phi(y) + ct^2\psi(y) + ct\nabla\phi(y) \cdot (y-x)}{(c^2t^2 - |y-x|^2)^{1/2}}\, dy \\ &= \frac{1}{2\pi c^2t^2} \int_{B(x,ct)} \frac{ct^2|y|^2}{(c^2t^2 - |y-x|^2)^{1/2}}\, dy \\ &= \frac{1}{2} ⨍_{B(x,ct)} \frac{ct^2|y|^2}{(c^2t^2 - |y-x|^2)^{1/2}}\, dy \end{aligned}$$

Now making the change of variables $y = x + ctz$, we have

$$\begin{aligned} u(x,t) &= \frac{1}{2} \fint_{B(0,1)} \frac{ct^2|x+ctz|^2}{(c^2t^2 - |ctz|^2)^{1/2}}\,dz \\ &= \frac{1}{2} \fint_{B(0,1)} \frac{ct^2|x+ctz|^2}{ct(1-|z|^2)^{1/2}}\,dz \\ &= \frac{t}{2} \fint_{B(0,1)} \frac{|x|^2 + 2ctx\cdot z + c^2t^2|z|^2}{(1-|z|^2)^{1/2}}\,dz. \end{aligned}$$

Now the first term can be evaluated as follows.

$$\begin{aligned} \frac{t|x|^2}{2\pi} \int_{B(0,1)} \frac{1}{(1-|z|^2)^{1/2}}\,dz &= \frac{t|x|^2}{2\pi} \int_0^{2\pi} \int_0^1 \frac{r}{(1-r^2)^{1/2}}\,dr\,d\theta \\ &= t|x|^2. \end{aligned}$$

For the second term, using the fact that $z_1/(1-|z|^2)^{1/2}$ is odd with respect to the z_2 axis (and similarly $z_2/(1-|z|^2)^{1/2}$ is odd with respect to the z_1 axis), we conclude that the second term is zero.

For the last term, we evaluate as follows,

$$\begin{aligned} \frac{c^2t^3}{2\pi} \int_{B(0,1)} \frac{|z|^2}{(1-|z|^2)^{1/2}}\,dS(z) &= \frac{c^2t^3}{2\pi} \int_0^{2\pi} \int_0^1 \frac{r^3}{(1-r^2)^{1/2}}\,dr\,d\theta \\ &= \frac{2c^2t^3}{3}. \end{aligned}$$

Therefore, our solution is given by

$$\boxed{u(x,t) = t|x|^2 + \frac{2c^2t^3}{3}.}$$

4. Solve

$$\begin{cases} U_t + AU_x = 0 \\ U(x,0) = \Phi(x) \end{cases}$$

where

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

and

$$\Phi(x) = \begin{bmatrix} \sin(x) \\ 1 \\ e^2 \end{bmatrix}.$$

Answer: First, we diagonalize our matrix A by looking for our eigenvalues. We look at $\det(A - \lambda I) = 0$. We see that our eigenvalues are given by $\lambda = 0, 2$. First, for

$\lambda_1 = 0$, we see that

$$A - \lambda_1 I = A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, an eigenvector associated with $\lambda_1 = 0$ is given by $\mathbf{v_1} = [-1\,1\,0]^T$. Then for $\lambda_2 = 2$, we have

$$A - \lambda_2 I = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we have two linearly independent eigenvectors, $\mathbf{v_2} = [1\,1\,0]^T$ and $\mathbf{v_3} = [0\,0\,1]^T$. Consequently, letting

$$Q = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\Lambda = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

we see that

$$Q^{-1} A Q = \Lambda.$$

In particular, plugging $A = Q\Lambda Q^{-1}$ into our equation, we have

$$U_t + Q\Lambda Q^{-1} U_x = 0.$$

Multiplying the equation by Q^{-1}, we have

$$Q^{-1} U_t + \Lambda Q^{-1} U_x = 0.$$

Then, letting $V = Q^{-1}U$, we have the decoupled initial-value problem

$$\begin{cases} V_t + \Lambda V_x = 0 \\ V(x,0) = \widetilde{\Psi}(x) \end{cases}$$

where

$$\widetilde{\Psi}(x) = Q^{-1}\widetilde{\Psi}(x) = \frac{1}{2}\begin{bmatrix} -\sin(x) + 1 \\ \sin(x) + 1 \\ 2e^2 \end{bmatrix}.$$

We have three linear transport equations. First,

$$\begin{cases} (v_1)_t = 0 \\ v_1(x,0) = \dfrac{1}{2}(-\sin(x) + 1) \end{cases}$$

implies

$$v_1(x,t) = \frac{1}{2}(-\sin(x) + 1).$$

Second,

$$\begin{cases} (v_2)_t + 2(v_2)_x = 0 \\ v_2(x,0) = \frac{1}{2}(\sin(x) + 1) \end{cases}$$

implies

$$v_2(x,t) = \frac{1}{2}(\sin(x-2t) + 1).$$

Last,

$$\begin{cases} (v_3)_t + 2(v_3)_x = 0 \\ v_3(x,0) = e^2 \end{cases}$$

implies

$$v_3(x,t) = e^2.$$

Therefore,

$$V = \frac{1}{2}\begin{bmatrix} -\sin(x) + 1 \\ \sin(x-2t) + 1 \\ 2e^2 \end{bmatrix}$$

and $U = QV$ implies our solution is given by

$$\boxed{U(x,t) = \frac{1}{2}\begin{bmatrix} \sin(x) + \sin(x-2t) \\ -\sin(x) + \sin(x-2t) + 2 \\ 2e^2 \end{bmatrix}}.$$

5. Consider the symmetric hyperbolic system

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_t + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{x_1} + \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_{x_2} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

   (a) Find the smallest ball in $\mathbb{R}^2$ in which the domain of dependence of $U(3,4,10)$ will lie. That is, find M such that the value of U at the point $(3,4,10)$ depends at most on the value of the initial data $U(x_1,x_2,0)$ in the ball of radius $10M$ about $(3,4)$.

   Answer: For a symmetric hyperbolic system, the domain of dependence of the solution at a point $(\vec{x}_0, t_0)$ is contained within the ball of radius Mt_0 where

$$M = \max_{|\vec{\xi}|=1,\ i=1,\ldots,m} |\lambda_i(\vec{\xi})|.$$

   where $\lambda_i(\xi)$, $i = 1,\ldots,m$ are the m eigenvalues of the matrix

$$A(\xi) = \sum_i^m \xi_i A_i.$$

Here, for $\xi \in \mathbb{R}^2$,

$$\begin{aligned} A(\xi) &= \xi_1 A_1 + \xi_2 A_2 \\ &= \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & 2\xi_1 \end{bmatrix}. \end{aligned}$$

The eigenvalues of $A(\xi)$ are given by

$$\lambda_1 = \frac{3\xi_1 + \sqrt{\xi_1^2 + 4\xi_2^2}}{2}$$
$$\lambda_2 = \frac{3\xi_1 - \sqrt{\xi_1^2 + 4\xi_2^2}}{2}$$

Now

$$\max_{|\vec{\xi}|=1} |\lambda_i(\vec{\xi})| = 2, \qquad i = 1, 2.$$

This can be found by maximizing λ_1, λ_2 subject to the constraint $|\vec{\xi}| = 1$. Therefore, $M = 2$, and consequently, the domain of dependence for the point $(3, 4, 10)$ is the ball $\{(x, y) \in \mathbb{R}^2 : |x - 3|^2 + |y - 4|^2 \leq (2 \cdot 10)^2\}$.

(b) Show that the ball you found in part (a) is the smallest ball in which you can guarantee the domain of dependence will lie, by showing there exists a direction $\xi = (\xi_1, \xi_2)$, where $|\xi| = 1$ for which there exists a plane wave solution $U(x_1, x_2, t) = V(x \cdot \xi - Mt)$; that is, a plane wave solution which travels at speed M. You don't need to calculate the plane wave solution.

Answer: From part (a), we see $\max_{|\vec{\xi}|=1} |\lambda_1(\vec{\xi})|$ occurs at $\vec{\xi} = (1, 0)$, in which case $\lambda_1(1, 0) = 2$. Similarly, $\max_{|\vec{\xi}|=1} |\lambda_2(\vec{\xi})|$ occurs at $\vec{\xi} = (-1, 0)$, in which case $\lambda_2(\vec{\xi}) = -2$. As this is a symmetric hyperbolic equation, we know there are m plane wave solutions for every direction $\xi \in \mathbb{R}^2$. In particular, for each $\xi \in \mathbb{R}^2$, the m plane wave solutions have speed $\lambda_i(\xi)$ for $i = 1, \ldots, m$.

To show this explicitly, in our case above, we look for a plane wave solution $\vec{v}(\vec{\xi} \cdot \vec{x} - \sigma t)$ where $\vec{\xi} = (1, 0)$. In other words, we are looking for a solution of the form $\vec{v}(x_1 - \sigma t)$ for some σ. Plugging this into our system, we have

$$-\sigma \vec{v}'(x_1 - \sigma t) + \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{v}'(x_1 - \sigma t) = \vec{0}.$$

This means we need to look for a function $\vec{v}'(x_1 - \sigma t)$ and a value σ such that

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \vec{v}'(x_1 - \sigma t) = \sigma \vec{v}'(x_1 - \sigma t).$$

In other words, an eigenvector $\vec{v}'$ and a corresponding eigenvalue σ for

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}.$$

Clearly, 2 is an eigenvalue of this matrix with corrsponding eigenvector $\vec{r}_2$. Letting $\vec{v}'(x_1 - \sigma t) = \vec{r}_2$, we see we have found a plane wave solution which travels in the direction $(1, 0)$ with speed 2.

(c) Find two plane wave solutions which propagate in the direction $(\xi_1, \xi_2) = (3/5, 4/5)$; that is, find two general solutions of the form $V_1(\xi \cdot x - \sigma_1 t)$, $V_2(\xi \cdot x - \sigma_2 t)$.

Answer: We look for a plane wave solution $\vec{v}(\vec{\xi} \cdot \vec{x} - \sigma t)$ which travels in the direction $\vec{\xi} = (3/5, 4/5)$. We plug

$$\vec{v}(\vec{\xi} \cdot \vec{x} - \sigma t) = \vec{v}(\xi_1 x_1 + \xi_2 x_2 - \sigma t)$$

into our system. Doing so, we have

$$-\sigma \vec{v}' + \xi_1 A_1 \vec{v}' + \xi_2 A_2 \vec{v}' = \vec{0},$$

where $\vec{v}' = \vec{v}'(\xi_1 x_1 + \xi_2 x_2 - \sigma t)$

$$A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Therefore, we need to look for an eigenvector $\vec{v}'$ and a corresponding eigenvalue σ of

$$A(\vec{\xi}) = \xi_1 A_1 + \xi_2 A_2$$

at $\vec{\xi} = (3/5, 4/5)$. Now

$$A(\vec{\xi}) = \begin{bmatrix} \xi_1 & \xi_2 \\ \xi_2 & 2\xi_1 \end{bmatrix} = \left(\frac{1}{5}\right) \begin{bmatrix} 3 & 4 \\ 4 & 6 \end{bmatrix}.$$

The eigenvalues are given by

$$\lambda_1 = \frac{1}{10}(9 + \sqrt{73})$$
$$\lambda_2 = \frac{1}{10}(9 - \sqrt{73})$$

with corresponding eigenvectors

$$\vec{r}_1 = \begin{bmatrix} \frac{1}{8}(-3 + \sqrt{73}) \\ 1 \end{bmatrix}$$
$$\vec{r}_2 = \begin{bmatrix} \frac{1}{8}(-3 - \sqrt{73}) \\ 1 \end{bmatrix}$$

Therefore, any function $\vec{v}_1'(\vec{\xi} \cdot \vec{x} - \lambda_1 t)$ which is a multiple of $\vec{r}_1$ will be an eigenfunction of $A(\vec{\xi})$, and, therefore, a plane wave solution. (Similarly, any function $\vec{v}_2'(\vec{\xi} \cdot \vec{x} - \lambda_2 t)$ which is a multiple of $\vec{r}_2$.) Therefore, two general plane wave solutions in the direction $\vec{\xi} = (3/5, 4/5)$ with speeds λ_1 and λ_2 given above, are of the form

$$\vec{v}_1(\vec{\xi} \cdot \vec{x} - \lambda_1 t) = f(\vec{\xi} \cdot \vec{x} - \lambda_1 t)\vec{r}_1,$$

and

$$\vec{v}_2(\vec{\xi} \cdot \vec{x} - \lambda_2 t) = g(\vec{\xi} \cdot \vec{x} - \lambda_2 t)\vec{r}_2$$

for arbitrary functions f and g, where $\vec{r}_1$ and $\vec{r}_2$ are given above.