7 Calculus of Variations

Ref: Evans, Sections 8.1, 8.2, 8.4

7.1 Motivation

The calculus of variations is a technique in which a partial differential equation can be reformulated as a minimization problem. In the previous section, we saw an example of this technique. Letting v_i denote the eigenfunctions of

$$(*)\begin{cases} -\Delta v = \lambda v & x \in \Omega \\ v = 0 & x \in \partial\Omega, \end{cases}$$

and defining the class of functions

$$Y_n = \{w \in C^2(\Omega), w \not\equiv 0, w = 0 \text{ for } x \in \partial\Omega, < w, v_i >= 0, i = 1, \ldots, n-1\},$$

we saw that if $u \in Y_n$ is a minimizer of the functional

$$I(w) = \frac{||\nabla w||^2_{L^2(\Omega)}}{||w||^2_{L^2(\Omega)}}$$

over Y_n, then u is an eigenfunction of (*) with corresponding eigenvalue

$$m \equiv \frac{||\nabla u||^2_{L^2(\Omega)}}{||u||^2_{L^2(\Omega)}}.$$

Further, we showed that m is in fact the n^{th} eigenvalue of (*).

In other words to find a solution of an eigenvalue problem, we reformulated the problem in terms of minimizing a certain functional. Proving the existence of an eigenfunction is now equivalent to proving the existence of a minimizer of I over the class Y_n. Proving the existence of a minimizer requires more sophisticated functional analysis. We will return to this idea later.

The example above could be reformulated equivalently to say that we are trying to minimize the functional

$$\widetilde{I}(w) = ||\nabla w||^2_{L^2(\Omega)}$$

over all functions $w \in Y_n$ such that $||w||^2_{L^2(\Omega)} = 1$. In particular, if v is in Y_n, then the normalized function $\widetilde{v} \equiv v/||v||^2_{L^2(\Omega)}$ (which has the same Rayleigh quotient as v) is in Y_n, and, of course, $||\widetilde{v}||^2_{L^2(\Omega)} = 1$. Therefore, minimizing I over functions $w \in Y_n$ is equivalent to minimizing $\widetilde{I}$ over functions $w \in Y_n$ subject to the constraint $||w||^2_{L(\Omega)} = 1$. This type of minimization problem is called a *constrained* minimization problem.

We begin by considering a simple example of how a partial differential equation can be rewritten as a minimizer of a certain functional over a certain class of admissible functions.

7.2 Dirichlet's Principle

In this section, we show that the solution of Laplace's equation can be rewritten as a minimization problem. Let

$$\mathcal{A} \equiv \{w \in C^2(\Omega), w = g \text{ for } x \in \partial\Omega\}.$$

Let

$$I(w) \equiv \frac{1}{2}\int_\Omega |\nabla w|^2\,dx.$$

Theorem 1. (Dirichlet's Principle) *Let Ω be an open, bounded subset of $\mathbb{R}^n$. Consider Laplace's equation on Ω with Dirichlet boundary conditions,*

$$(*)\begin{cases} \Delta u = 0 & x \in \Omega \\ u = g & x \in \partial\Omega. \end{cases}$$

The function $u \in \mathcal{A}$ is a solution of (*) if and only if

$$I(u) = \min_{w\in\mathcal{A}} I(w).$$

Proof. First, we suppose u is a solution of (*). We need to show that $I(u) \leq I(w)$ for all $w \in \mathcal{A}$. Let $w \in \mathcal{A}$. Then

$$\begin{aligned} 0 &= \int_\Omega \Delta u(u - w)\,dx \\ &= -\int_\Omega |\nabla u|^2\,dx + \int_\Omega \nabla u \cdot \nabla w\,dx \\ &\leq -\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega [|\nabla u|^2 + |\nabla w|^2]\,dx \\ &= -\frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2}\int_\Omega |\nabla w|^2\,dx. \end{aligned}$$

Therefore,

$$\int_\Omega |\nabla u|^2\,dx \leq \int_\Omega |\nabla w|^2\,dx.$$

But w is an arbitrary function in $\mathcal{A}$. Therefore,

$$I(u) = \min_{w\in\mathcal{A}} I(w).$$

Next, suppose u minimizes I over all $w \in \mathcal{A}$. We need to show that u is a solution of (*). Let v be a C^2 function such that $v \equiv 0$ for $x \in \partial\Omega$. Therefore, for all ϵ, $u + \epsilon v \in \mathcal{A}$. Now let

$$i(\epsilon) \equiv I(u + \epsilon v).$$

By assumption, u is a minimizer of I. Therefore, i must have a minimum at $\epsilon = 0$, and, therefore, $i'(0) = 0$. Now

$$\begin{aligned} i(\epsilon) &= I(u + \epsilon v) \\ &= \int_\Omega |\nabla(u + \epsilon v)|^2\,dx \\ &= \int_\Omega [|\nabla u|^2 + 2\epsilon\nabla u \cdot \nabla v + \epsilon^2|\nabla v|^2]\,dx, \end{aligned}$$

implies

$$i'(\epsilon) = \int_\Omega [\nabla u \cdot \nabla v] + 2\epsilon |\nabla v|^2 \, dx.$$

Therefore,

$$\begin{aligned} i'(0) &= \int_\Omega \nabla u \cdot \nabla v \, dx \\ &= -\int_\Omega (\Delta u) v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v \, dS(x) \\ &= -\int_\Omega (\Delta u) v \, dx. \end{aligned}$$

Now $i'(0) = 0$ implies

$$\int_\Omega (\Delta u) v \, dx = 0.$$

Since this is true for all $v \in C^2(\Omega)$ such that $v = 0$ for $x \in \partial\Omega$, we can conclude that $\Delta u = 0$, as claimed. □

7.3 Euler-Lagrange Equations

Laplace's equation is an example of a class of partial differential equations known as Euler-Lagrange equations. These equations are defined as follows. Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let L be a smooth function such that

$$L : \mathbb{R}^n \times \mathbb{R} \times \overline{\Omega} \to \mathbb{R}.$$

We will write $L = L(p, z, x)$ where $p \in \mathbb{R}^n$, $z \in \mathbb{R}$ and $x \in \overline{\Omega}$. Associated with a function L, we define the **Euler-Lagrange equation**

$$-\sum_{i=1}^n (L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0.$$

The function L is known as the **Lagrangian**.

Example 2. Let

$$L(p, z, x) = \frac{1}{2}|p|^2.$$

The associated Euler-Lagrange equation is just Laplace's equation

$$\Delta u = 0.$$

◇

Example 3. Let

$$L(p, z, x) = \frac{1}{2} \sum_{i,j=1}^n a^{ij}(x) p_i p_j - z f(x)$$

where $a^{ij} = a^{ji}$. The associated Euler-Lagrange equation is

$$-\sum_{i,j=1}^{n} (a^{ij} u_{x_j})_{x_i} = f,$$

a generalization of Poisson's equation. $\diamond$

Recall that Dirichlet's principle stated that a solution of

$$\begin{cases} \Delta u = 0 & x \in \Omega \subset \mathbb{R}^n \\ u = g & x \in \partial\Omega \end{cases}$$

is a minimizer of

$$I(w) = \frac{1}{2} \int_\Omega |\nabla w|^2 \, dx$$

over C^2 functions which satisfy the boundary condition. In other words, a harmonic function on Ω is a minimizer of

$$I(w) = \int_\Omega L(\nabla w, w, x) \, dx$$

where L is the associated Lagrangian given by $L(p, z, x) = \frac{1}{2}|p|^2$.

For a given Lagrangian L define

$$I_L(w) = \int_\Omega L(\nabla w, w, x) \, dx.$$

We will now show that if u is a minimizer of $I_L(w)$ over an admissible class of functions $\mathcal{A}$, then u is a solution of the associated Euler-Lagrange equation

$$-\sum_{i=1}^{n} (L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) = 0.$$

As in the proof of Dirichlet's principle, suppose u is a minimizer of

$$I_L(w) = \int_\Omega L(\nabla w, w, x) \, dx$$

over an admissible class of functions $\mathcal{A}$. Let $v \in C^\infty(\Omega)$ such that v has compact support within Ω. We denote this space of functions by $C_c^\infty(\Omega)$. Define

$$i(\epsilon) = I(u + \epsilon v).$$

If u is a minimizer of I, then $i'(0) = 0$.

$$\begin{aligned} i(\epsilon) &= I(u + \epsilon v) \\ &= \int_\Omega L(\nabla u + \epsilon \nabla v, u + \epsilon v, x) \, dx. \end{aligned}$$

Therefore,

$$i'(\epsilon) = \int_\Omega \sum_{i=1}^n L_{p_i}(\nabla u + \epsilon \nabla v, u + \epsilon v, x) v_{x_i} + L_z(\nabla u + \epsilon \nabla v, u + \epsilon v, x) v \, dx.$$

Now $i'(0) = 0$ implies

$$0 = i'(0) = \int_\Omega \sum_{i=1}^n L_{p_i}(\nabla u, u, x) v_{x_i} + L_z(\nabla u, u, x) v \, dx.$$

Integrating by parts and using the fact that $v = 0$ for $x \in \partial\Omega$, we conclude that

$$\int_\Omega \left[-\sum_{i=1}^n (L_{p_i}(\nabla u, u, x))_{x_i} + L_z(\nabla u, u, x) \right] v \, dx = 0.$$

Since this is true for all $v \in C_c^\infty(\Omega)$, we conclude that u is a solution of the Euler-Lagrange equation associated with the Lagrangian L. Consequently, to solve Euler-Lagrange equations, we can reformulate these partial differential equations as minimization problems of the functionals

$$I_L(w) = \int_\Omega L(\nabla w, w, x) \, dx.$$

Above we showed how solving certain partial differential equations could be rewritten as minimization problems. Sometimes, however, the minimization problem is the physical problem which we are interested in solving.

Example 4. (Minimal Surfaces) Let $w : \Omega \to \mathbb{R}$. The surface area of the graph of w is given by

$$I(w) = \int_\Omega (1 + |\nabla w|^2)^{1/2} \, dx.$$

The problem is to look for the *minimal surface*, the surface with the least surface area, which satisfies the boundary condition $w = g$ for $x \in \partial\Omega$. Alternatively, this minimization problem can be written as a partial differential equation. In particular, the Lagrangian associated with I is

$$L(p, z, x) = (1 + |p|^2)^{1/2}.$$

The associated Euler-Lagrange equation is

$$\sum_{i=1}^n \left(\frac{u_{x_i}}{(1 + |\nabla u|^2)^{1/2}} \right)_{x_i} = 0.$$

This equation is known as the *minimal surface equation.* ◇

7.4 Existence of Minimizers

We now discuss the existence of a minimizer of

$$I(w) = \int_\Omega L(\nabla w, w, x) \, dx$$

over some admissible class of functions $\mathcal{A}$. We will discuss existence under two assumptions on the Lagrangian L: *convexity* and *coercivity.* We discuss these issues now.

7.4.1 Convexity

Assume u is a minimizer of

$$I(w) = \int_\Omega L(\nabla w, w, x)\, dx.$$

We discussed earlier that if u is a minimizer of I, then for any $v \in C_c^\infty(\Omega)$, the function

$$i(\epsilon) = I(u + \epsilon v)$$

has a local minimum at $\epsilon = 0$, and, therefore, $i'(0) = 0$. In addition, if i has a minimum at $\epsilon = 0$, then $i''(0) \geq 0$. We now calculate $i''(0)$ explicitly to see what this implies about I and L.

By a straightforward calculation, we see that

$$\begin{aligned} i''(\epsilon) = \int_\Omega \Bigg[& \sum_{i,j=1}^n L_{p_i p_j}(\nabla u + \epsilon \nabla v, u + \epsilon v, x) v_{x_i} v_{x_j} \\ & + 2 \sum_{i=1}^n L_{p_i z}(\nabla u + \epsilon \nabla v, u + \epsilon v, x) v_{x_i} v + L_{zz}(\nabla u + \epsilon \nabla v, u + \epsilon v, x) v^2 \Bigg]\, dx. \end{aligned}$$

Therefore, we conclude that

$$\begin{aligned} 0 \leq i''(0) = \int_\Omega \Bigg[& \sum_{i,j=1}^n L_{p_i p_j}(\nabla u, u, x) v_{x_i} v_{x_j} + 2 \sum_{i=1}^n L_{p_i z}(\nabla u, u, x) v_{x_i} v \\ & + L_{zz}(\nabla u, u, x) v^2 \Bigg]\, dx \end{aligned} \tag{7.1}$$

for all $v \in C_c^\infty(\Omega)$. By an approximation argument, we can show that (7.1) is also valid for the function

$$v(x) = \delta \rho \left(\frac{x \cdot \xi}{\delta} \right) \zeta(x)$$

where $\delta > 0$, $\xi \in \mathbb{R}^n$, $\zeta \in C_c^\infty(\Omega)$ and $\rho : \mathbb{R} \to \mathbb{R}$ is the periodic "zig-zag" function

$$\rho(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ 1 - x & \frac{1}{2} \leq x \leq 1 \end{cases}$$

and $\rho(x+1) = \rho(x)$ elsewhere. Notice that the function ρ satisfies $|\rho'| = 1$ "almost everywhere" (except on a set of measure zero). Calculating the partial derivative of v with respect to x_i, we see that

$$v_{x_i}(x) = \rho' \left(\frac{x \cdot \xi}{\delta} \right) \xi_i \zeta(x) + O(\delta)$$

as $\delta \to 0$. Combining this fact with (7.1), using the fact that $|\rho'| = 1$ almost everywhere, and taking the limit as $\delta \to 0$, we conclude that

$$0 \leq \int_\Omega \sum_{i,j=1}^n L_{p_i p_j}(\nabla u, u, x) \xi_i \xi_j \zeta \, dx.$$

Since this estimate holds for all $\zeta \in C_c^\infty(\Omega)$, we conclude that

$$0 \leq \sum_{i,j=1}^n L_{p_i p_j}(\nabla u, u, x)\xi_i \xi_j.$$

This is a necessary condition for a minimizer. Therefore, to guarantee the possibility of a minimizer, we will assume the following **convexity condition** on the Lagrangian L,

$$\sum_{i,j=1}^n L_{p_i p_j}(p, z, x)\xi_i \xi_j \geq 0. \tag{7.2}$$

That is, we will assume L is convex in the variable p.

7.4.2 Coercivity

In order to prove that the functional I has a minimum, we would need to know that I is bounded below. Of course, this is not enough to guarantee the existence of a minimizer. For example, the function $f(x) = e^{-x^2}$ is bounded below by zero. In addition, $\inf_{x \in \mathbb{R}} f(x) = 0$. However, the infimum is never achieved. One way of guaranteeing that a continuous function $f : \mathbb{R} \to \mathbb{R}$ achieves its infimum is to assume that $f(x) \to +\infty$ as $|x| \to +\infty$.

Using this same idea for our functional I, we will assume that $I(w) \to +\infty$ as $|w| \to +\infty$. In particular, we will assume the following **coercivity condition**. Let $1 < q < \infty$ be fixed.

$$\exists\ \alpha > 0, \beta \geq 0 \text{ s.t. } L(p, z, x) \geq \alpha |p|^q - \beta \quad \forall\ (p, z, x) \in \mathbb{R}^n \times \mathbb{R} \times \Omega. \tag{7.3}$$

If the Lagrangian L satisfies this coercivity condition, then the functional I satisfies

$$\begin{aligned} I(w) &= \int_\Omega L(\nabla w, w, x)\, dx \\ &\geq \int_\Omega \alpha |\nabla w|^q \, dx - \beta|\Omega|. \end{aligned}$$

Therefore, $I(w) \to +\infty$ as $\int_\Omega |\nabla w|^q \, dx \to +\infty$.

7.4.3 Theorem on Existence of Minimizers

In this section, we state an existence result for minimizers. We will be considering the functional

$$I(w) = \int_\Omega L(\nabla w, w, x)\, dx$$

where the Lagrangian L satisfies the convexity and coercivity conditions described in the previous sections. We will prove the existence of a minimizer u of I over a certain class of admissible functions $\mathcal{A}$. We describe this class of functions now. First, for $1 \leq q < \infty$, let

$$L^q(\Omega) \equiv \{u : \int_\Omega |u|^q \, dx < \infty\}.$$

We define the norm of a function $u \in L^q(\Omega)$ as

$$||u||_{L^q(\Omega)} = \left(\int_\Omega |u|^q \, dx \right)^{1/q}.$$

Now we let

$$W^{1,q}(\Omega) \equiv \{u : u \in L^q(\Omega), \nabla u \in L^q(\Omega)\}.$$

Remark. The space $W^{1,q}$ is an example of a Sobolev space. For a more thorough introduction to Sobolev spaces, see Evans, Chapter 5.

Now we let

$$\mathcal{A} \equiv \{w \in W^{1,q}(\Omega) | \, w = g \text{ on } \partial\Omega\}.$$

Theorem 5. *Assume L satisfies the coercivity condition* (7.3) *and the convexity condition* (7.2)*. Assume the set $\mathcal{A}$ is nonempty. Then there exists at least one function $u \in \mathcal{A}$ such that*

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Below, we give an outline of the proof. The complete details rely on functional analysis facts which are beyond the scope of this course.

Outline of Proof.

1. By the coercivity assumption (7.3), we know that I is bounded below. Therefore, I has an infimum. Let

   $$m \equiv \int_{w \in \mathcal{A}} I(w).$$

   If $m = \infty$, then we're done. Therefore, we assume that m is finite. Let u_k be a *minimizing sequence*. That is, assume $I(u_k) \to m$ as $k \to +\infty$. We now want to show that there exists a $u \in \mathcal{A}$ such that $I(u) = m$. First, we will show that u_k is a bounded sequence in $W^{1,q}$. This will imply there exists a $u \in W^{1,q}$ such that $u_k \rightharpoonup u$ (converges weakly). Then we need to show that $u \in \mathcal{A}$ and $I(u) = m$.

2. Assume $\beta = 0$ in the coercivity assumption (7.3). (Otherwise, we could consider $\widetilde{L} = L + \beta$.) Therefore,

   $$L(p, z, x) \geq \alpha |p|^q.$$

   which implies

   $$I(w) \geq \alpha \int_\Omega |\nabla w|^q \, dx.$$

   Therefore, for the minimizing sequence u_k,

   $$I(u_k) \geq \alpha \int_\Omega |\nabla u_k|^q \, dx.$$

   Now $I(u_k) \to m$ which is finite implies that

   $$\sup_k \int_\Omega |\nabla u_k|^q \, dx < \infty.$$

3. Fix any $w \in \mathcal{A}$. Since $u_k = g = w$ on $\partial\Omega$, $u_k - w = 0$ for $x \in \partial\Omega$. Therefore, we can use Poincare's inequality. Therefore, we have

$$\begin{aligned} ||u_k||_{L^q(\Omega)} &\leq ||u_k - w||_{L^q(\Omega)} + ||w||_{L^q(\Omega)} \\ &\leq C||\nabla u_k - \nabla w||_{L^q(\Omega)} + C \leq C. \end{aligned}$$

   Therefore,

$$\sup_k ||u_k||_{L^q(\Omega)} < +\infty.$$

   That is, u_k is a bounded sequence in $W^{1,q}(\Omega)$. Fact: This implies there exists a function $u \in W^{1,q}(\Omega)$ and a subsequence $\{u_{k_j}\}$ such that u_{k_j} converges to u weakly, meaning

$$\int_\Omega (u_{k_j} - u)v\,dx \to 0$$

   as $k \to +\infty$ for all v in the *dual space* of $W^{1,q}(\Omega)$. We write this as $u_{k_j} \rightharpoonup u$.

4. Now we need to show that $u \in \mathcal{A}$. From the previous step, one can show that $u \in W^{1,q}(\Omega)$. Therefore, it just remains to show that u satisfies the necessary boundary conditions. *See Evans, Sec. 8.2.*

5. Using the fact that $u_{k_j} \rightharpoonup u$, and the fact that L is convex and bounded below, we are able to conclude that

$$I(u) \leq \liminf I(u_{k_j}) = m.$$

   See Evans, Sec. 8.2. But, since $u \in \mathcal{A}$, $I(u) \geq m$. Therefore, $I(u) = m$.

□

7.5 Constrained Minimization Problems

In this section, we discuss constrained minimization problems: minimizing a functional $I(w)$ subject to the constraint $J(w) = 0$.

Example 6. Earlier we mentioned that the Minimum Principle for the n^{th} Eigenvalue could be written as a constrained minimization problem. Let

$$J(w) \equiv ||w||^2_{L^2(\Omega)}.$$

Let

$$\mathcal{A} \equiv \{w \in Y_n | J(w) = 0\}.$$

Let

$$I(w) = ||\nabla w||^2_{L^2(\Omega)}.$$

We showed that if $u \in A$ satisfies

$$I(u) = \min_{w \in \mathcal{A}} I(w),$$

then u satisfies

$$\begin{cases} -\Delta u = \lambda_n u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where

$$\lambda_n = ||\nabla u||^2_{L^2(\Omega)}.$$

We will now state and prove a generalization of the result in the above example. Let

$$I(w) \equiv \frac{1}{2}\int_\Omega |\nabla w|^2\,dx.$$

Let $G : \mathbb{R} \to \mathbb{R}$ be a smooth function. Let $g = G'$. Let

$$J(w) \equiv \int_\Omega G(w(x))\,dx.$$

Let

$$\mathcal{A} \equiv \{w \in C^2(\Omega), w = 0 \text{ for } x \in \partial\Omega |\ J(w) = 0\}.$$

Theorem 7. *Suppose there exists a $u \in \mathcal{A}$ such that*

$$I(u) = \min_{w \in \mathcal{A}} I(w).$$

Then there exists a real number λ such that

$$\begin{cases} -\Delta u = \lambda g(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

The number λ is called the **Lagrange multiplier** corresponding to the integral constraint $J(u) = 0$.

Proof. We would like to use a similar argument as used in the unconstrained minimization problem. In particular, we would like to find w such that $u + \epsilon w \in \mathcal{A}$, and then look at $i(\epsilon) = I(u + \epsilon w)$. Unlike in the previous case, however, we cannot just choose $w \in C_c^\infty(\Omega)$. Because, this does not necessarily imply that $u + \epsilon w$ is in $\mathcal{A}$. We construct such a w as follows.

Let $v \in C_c^\infty(\Omega)$. As mentioned above $u + \epsilon v$ is not necessarily in $\mathcal{A}$, because it does not necessarily satisfy the constraint $J(w) = 0$. But, maybe we could find some function $w \in C^2(\Omega)$ which would "correct" this problem. Maybe we could find some w and some function $\phi(\epsilon)$ such that $u + \epsilon v + \phi(\epsilon) w \in \mathcal{A}$. Choose w such that

$$\int_\Omega g(u) w\,dx \neq 0. \tag{7.4}$$

Define

$$j(\epsilon, \sigma) \equiv J(u + \epsilon v + \sigma w) = \int_\Omega G(u + \epsilon v + \sigma w)\,dx.$$

We see that

$$j(0,0) = \int_\Omega G(u)\,dx = 0,$$

by the assumption that $u \in \mathcal{A}$. We see that

$$\frac{\partial j}{\partial \sigma}(\epsilon, \sigma) = \int_\Omega g(u + \epsilon v + \sigma w) w \, dx.$$

Now by the assumption (7.4), we see that

$$\frac{\partial j}{\partial \sigma}(0, 0) \neq 0.$$

Therefore, by the implicit function, there exists a C^1 function $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$\phi(0) = 0$$

and

$$j(\epsilon, \phi(\epsilon)) = 0.$$

In particular, this implies that

$$j(\epsilon, \phi(\epsilon)) = J(u + \epsilon v + \phi(\epsilon) w) = 0,$$

which means that $u + \epsilon v + \phi(\epsilon) w \in \mathcal{A}$.

Now we can use a similar argument as to the one we used earlier. Let

$$i(\epsilon) = I(u + \epsilon v + \phi(\epsilon) w) = \frac{1}{2} \int_\Omega |\nabla u + \epsilon \nabla v + \phi(\epsilon) \nabla w|^2 \, dx.$$

By the assumption that u is a minimizer of I over all functions in $\mathcal{A}$, we know that $i'(0) = 0$. Now

$$i'(\epsilon) = \int_\Omega (\nabla u + \epsilon \nabla v + \phi(\epsilon) \nabla w) \cdot (\nabla v + \phi'(\epsilon) \nabla w) \, dx$$

implies

$$i'(0) = \int_\Omega (\nabla u \cdot \nabla v + \phi'(0) \nabla u \cdot \nabla w) \, dx = 0. \tag{7.5}$$

Using the fact that $j(\epsilon, \phi(\epsilon)) = 0$, we can calculate $\phi'(0)$ as follows.

$$\frac{\partial j}{\partial \epsilon}(\epsilon, \phi(\epsilon)) + \frac{\partial j}{\partial \sigma}(\epsilon, \phi(\epsilon)) \phi'(\epsilon) = 0$$

implies

$$\phi'(0) = -\frac{j_\epsilon(0,0)}{j_\sigma(0,0)} = -\frac{\int_\Omega g(u) v \, dx}{\int_\Omega g(u) w \, dx}. \tag{7.6}$$

Combining (7.5) with (7.6), we conclude that

$$\begin{aligned} \int_\Omega \nabla u \cdot \nabla v \, dx &= \frac{\int_\Omega g(u) v \, dx}{\int_\Omega g(u) w \, dx} \int_\Omega \nabla u \cdot \nabla w \, dx \\ &= \frac{\int_\Omega \nabla u \cdot \nabla w \, dx}{\int_\Omega g(u) w \, dx} \int_\Omega g(u) v \, dx. \end{aligned}$$

Defining

$$\lambda \equiv \frac{\int_\Omega \nabla u \cdot \nabla w \, dx}{\int_\Omega g(u) w \, dx},$$

we conclude that

$$\int_\Omega \nabla u \cdot \nabla v \, dx = \lambda \int_\Omega g(u) v \, dx.$$

Using the integration by parts formula on the left-hand side, and the fact that v vanishes for $x \in \partial\Omega$, we conclude that

$$-\int_\Omega \Delta u \, v \, dx = \lambda \int_\Omega g(u) v \, dx.$$

Since this is true for all $v \in C^2(\Omega)$ which vanish for $x \in \partial\Omega$, we conclude that

$$\begin{cases} -\Delta u = \lambda g(u) & x \in \Omega \\ u = 0 & x \in \partial\Omega. \end{cases}$$

$\square$